CMSE 890-001: Spectral Graph Theory and Related Topics, MSU, Spring 2021

Lecture 16: Graph Conductance

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25 Conductance

The conductance of a weighted graph $G = (V, E, w)$ provides and alternative measure to the isoperimetric ratio by which to measure its connectivity. We set some preliminary notation before defining the conductance. First, recall that in a weighted graph the degree of a vertex $a \in V$ is the sum of the weights of the edges connected to a:

$$\boldsymbol{d}(a) := \deg(a) := \sum_{b \in N(a)} w(a, b)\,.$$

For a subset $S \subseteq V$, define $d(S)$ as the sum of the degrees of all vertices in S:

$$d(S) := \sum_{a \in S} \deg(a) = \sum_{a \in S} \sum_{b \in N(a)} w(a, b)\,.$$

Notice that

$$d(V) = 2 \sum_{(a,b) \in E} w(a, b)\,.$$

For a subset $F \subseteq E$, define $w(F)$ to be the sum of the weights of the edges in F:

$$w(F) := \sum_{(a,b) \in F} w(a, b)\,.$$

Finally, recall that the boundary of S is:

$$\partial S := \{(a, b) \in E : a \in S \text{ and } b \in V - S\}\,.$$

We define the *conductance* of S as

$$\varphi(S) := \frac{w(\partial S)}{\min(d(S), d(V - S))}\,.$$

The conductance of the graph G minimizes $\varphi(S)$ over all subsets S:

$$\varphi_G := \min_{S \subset V} \varphi(S)\,.$$

In order to get a better feel for the conductance of a graph, it may be useful to consider the following calculation for $d(S)$:

$$\begin{aligned} d(S) &= \sum_{a \in S} \sum_{b \in N(a)} w(a,b) \\ &= \sum_{a \in S} \sum_{\substack{(a,b) \in E \\ b \in S}} w(a,b) + \sum_{a \in S} \sum_{\substack{(a,b) \in E \\ b \in V-S}} w(a,b) \\ &= 2w(E(S)) + w(\partial S)\,. \end{aligned} \tag{55}$$

Thus, if $\min(d(S), d(V-S)) = d(S)$, then

$$\varphi(S) = \frac{w(\partial S)}{2w(E(S)) + w(\partial S)}\,.$$

On the other hand, note that $w(\partial S) = w(\partial(V-S))$. Thus, if $\min(d(S), d(V-S)) = d(V-S)$, we have

$$\varphi(S) = \frac{w(\partial(V-S))}{2w(E(V-S)) + w(\partial(V-S))}\,.$$

These two facts also imply that

$$\varphi(S) \leq 1 \quad \text{and} \quad \varphi(S) = \varphi(V-S)\,.$$

Remark 24. Let us compare the isoperimetric ratio of a set S to its conductance for an unweighted graph $G = (V, E)$. Let us assume that $\min(d(S), d(V-S)) = d(S)$ for a bit of added simplicity. In this case we have

$$\theta(S) = \frac{|\partial S|}{|S|} \quad \text{and} \quad \varphi(S) = \frac{|\partial S|}{d(S)}\,.$$

Thus in both cases the numerator is the same, but it is the denominator that changes. In particular, the isoperimetric ratio, $\theta(S)$, places more importance on the number of vertices being removed if one were to remove $G(S) = (S, E(S))$ from G, as indicated by having $|S|$ in the denominator. On the other hand, the conductance places more importance on the number of edges being removed, since $d(S) = 2|E(S)| + |\partial S|$.

Remark 25. If $G = (V, E)$ is d-regular and $|S| \leq n/2$, then $\theta(S)$ and $\varphi(S)$ differ by a factor of d:

$$\varphi(S) = \frac{|\partial S|}{d(S)} = \frac{|\partial S|}{d|S|} = \frac{\theta(S)}{d}\,.$$

The above considerations indicate the conductance is a degree-invariant measure of connectivity of a weighted graph $G = (V, E, w)$. It thus makes sense to try to relate it to ν_2, the second eigenvalue of the normalized graph Laplacian $\boldsymbol{N}$, as opposed to λ_2, the second eigenvalue of the graph Laplacian $\boldsymbol{L}$. Our goal is to prove that

$$\frac{\nu_2}{2} \leq \varphi_G \leq \sqrt{2\nu_2}\,,$$

and to construct a set $S \subset V$ such that $\varphi(S) \leq \sqrt{2\nu_2}$. In doing so, we will have shown that ν_2 characterizes the connectivity of G, and furthermore, we will be able to turn the construction of S into an algorithm that allows us to compute a nearly optimal partition of G. The next theorem provides the lower bound for φ_G, which is an analogue of the similar result for the isoperimetric ratio given in Theorem 39.

Theorem 45. *Let $G = (V, E, w)$, let $0 = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n \leq 2$ be its normalized graph Laplacian eigenvalues, and let $S \subset V$. Then*

$$\frac{d(V)w(\partial S)}{d(S)d(V - S)} \geq \nu_2\,,$$

and as such,

$$\varphi(S) \geq \nu_2/2 \quad \implies \quad \varphi_G \geq \nu_2/2\,.$$

Proof. The proof is pretty similar to the proof of Theorem 39. By Theorem 44 we know that

$$\nu_2 = \min_{\substack{\boldsymbol{y} \in \mathbb{R}^n \\ \langle \boldsymbol{y}, \boldsymbol{d} \rangle = 0}} \frac{\boldsymbol{y}^T \boldsymbol{L} \boldsymbol{y}}{\boldsymbol{y}^T \boldsymbol{D} \boldsymbol{y}}\,,$$

and so we again want to use the test vector technique to bound ν_2 from above. We need a test vector that will give us the right quantities for the set S. As before, $\boldsymbol{y} = \boldsymbol{1}_S$ would be a nice choice, but it is not orthogonal to $\boldsymbol{d}$. So instead we select:

$$\boldsymbol{y} = \boldsymbol{1}_S - \sigma \boldsymbol{1}\,, \quad \sigma = \frac{d(S)}{d(V)}\,.$$

We check that $\langle \boldsymbol{y}, \boldsymbol{d} \rangle = 0$; indeed:

$$\langle \boldsymbol{y}, \boldsymbol{d} \rangle = (\boldsymbol{1}_S - \sigma \boldsymbol{1})^T \boldsymbol{d} = \boldsymbol{1}_S^T \boldsymbol{d} - \sigma \boldsymbol{1}^T \boldsymbol{d} = d(S) - \sigma d(V) = d(S) - \frac{d(S)}{d(V)} d(V) = 0\,.$$

Using a similar argument as in the proof of Theorem 39, let us compute:

$$\begin{aligned}
\boldsymbol{y}^T \boldsymbol{L} \boldsymbol{y} &= \sum_{(a,b) \in E} w(a,b)((\boldsymbol{1}_S(a) - \sigma) - (\boldsymbol{1}_S(b) - \sigma))^2 \\
&= \sum_{(a,b) \in E} w(a,b)(\boldsymbol{1}_S(a) - \boldsymbol{1}_S(b))^2 \\
&= \sum_{(a,b) \in \partial S} w(a,b) \\
&= w(\partial S)\,.
\end{aligned}$$

Now we compute the denominator:

$$\begin{aligned}
\boldsymbol{y}^T \boldsymbol{D} \boldsymbol{y} = \sum_{a \in V} \boldsymbol{d}(a) \boldsymbol{y}(a)^2 &= \sum_{a \in S} (1-\sigma)^2 \boldsymbol{d}(a) + \sum_{a \in V-S} (-\sigma)^2 \boldsymbol{d}(a) \\
&= (1-\sigma)^2 d(S) + \sigma^2 d(V-S) \\
&= d(S) - 2\sigma d(S) + \sigma^2 d(S) + \sigma^2 d(V-S) \\
&= d(S) - 2\sigma d(S) + \sigma^2 d(V) \\
&= d(S) - 2\sigma d(S) + \sigma \frac{d(S)}{d(V)} d(V) \\
&= d(S) - \sigma d(S) \\
&= \left(1 - \frac{d(S)}{d(V)}\right) d(S) \\
&= \frac{(d(V) - d(S)) d(S)}{d(V)} \\
&= \frac{d(V-S) d(S)}{d(V)} \,.
\end{aligned}$$

Putting together our computations for $\boldsymbol{y}^T \boldsymbol{L} \boldsymbol{y}$ and $\boldsymbol{y}^T \boldsymbol{D} \boldsymbol{y}$ we have the result:

$$\nu_2 \leq \frac{\boldsymbol{y}^T \boldsymbol{L} \boldsymbol{y}}{\boldsymbol{y}^T \boldsymbol{D} \boldsymbol{y}} = \frac{d(V) w(\partial S)}{d(V-S) d(S)} \,.$$

To complete the proof, note that since $d(V) = d(S) + d(V-S)$, we have:

$$\max(d(S), d(V-S)) \geq d(V)/2 \quad \Longrightarrow \quad \frac{d(V)}{\max(d(S), d(V-S))} \leq 2 \,.$$

Therefore,

$$\begin{aligned}
\nu_2 &\leq \frac{d(V) w(\partial S)}{d(S) d(V-S)} \\
&= \frac{d(V) w(\partial S)}{\max(d(S), d(V-S)) \min(d(S), d(V-S))} \\
&\leq \frac{2 w(\partial S)}{\min(d(S), d(V-S))} \\
&= 2\varphi(S) \,.
\end{aligned}$$

□

References

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