CMSE 890-001: Spectral Graph Theory and Related Topics, MSU, Spring 2021

Homework 07

Due: March 21, 2021

These first three exercises ask you to prove some things we stated, but did not prove, in class.

Exercise 1. Prove Theorem 38 from Lecture 14.

Exercise 2. Let $G = (V, E, w)$ be a graph with graph Laplacian eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Set $g_{\text{low}} : [0, \infty) \to \mathbb{R}$ to be any function such that $g_{\text{low}}(t) = 1$ for all $t \in [0, \lambda_n]$ and then $g_{\text{low}}(t)$ decreases to 0 for $t > \lambda_n$. Set

$$g_{\text{high}}(t) = \left[g_{\text{low}}(t)^2 - g_{\text{low}}(2t)^2 \right]^{1/2} .$$

For $J > 0$, prove that

$$|g_{\text{low}}(2^J t)|^2 + \sum_{j=0}^{J-1} |g_{\text{high}}(2^j t)|^2 = 1 , \quad \forall\, t \in [0, \lambda_n] . \tag{1}$$

Exercise 3. Let $G = (V, E, w)$ be a graph with no isolated vertices, and let $\boldsymbol{N}_G$ be its normalized graph Laplacian with eigenvalues $0 = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$. Prove that $\nu_n \leq 2$. *Hint: The inequality $(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2)$, for $\alpha, \beta \in \mathbb{R}$, may be useful.*

Now let us do some more programming work on graph signal processing.

Exercise 4 (20 points)**.** Using the bunny graph from the previous homework, implement the graph wavelet transform using your own choice of g_{low} and g_{high}. Note, you do not necessarily have to satisfy (1), but you should ensure that

$$|g_{\text{low}}(2^J t)|^2 + \sum_{j=0}^{J-1} |g_{\text{high}}(2^j t)|^2 \geq \delta > 0 , \quad \forall\, t \in [0, \lambda_n] , \tag{2}$$

for some $\delta > 0$. Turn in a Python Jupyter notebook in which you do the following (5 points for each item):

- On a single plot, plot the graph Fourier transform of your low pass, $\widehat{\boldsymbol{\ell}}_J(k)$, and the graph Fourier transforms of all your wavelets, $\widehat{\boldsymbol{h}}_j$ for $0 \leq j < J$, as a function of λ_k.
- In $J + 1$ separate plots, plot $(\boldsymbol{\ell}_J)_b$ and $(\boldsymbol{h}_j)_b$ overlaid on the bunny manifold using vertex $b = 1000$.

- For the signal

$$\boldsymbol{x}(a) = \begin{cases} 1 & \boldsymbol{\psi}_2(a) \geq 0 \\ 0 & \boldsymbol{\psi}_2(a) < 0 \end{cases} ,$$

  compute the wavelet transform $\boldsymbol{W}_J\boldsymbol{x} = \{\boldsymbol{x} * \boldsymbol{\ell}_J\,,\, \boldsymbol{x} * \boldsymbol{h}_j : 0 \leq j < J\}$. In $J+1$ separate plots, plot $\boldsymbol{x} * \boldsymbol{\ell}_J$ and $\boldsymbol{x} * \boldsymbol{h}_j$ for $0 \leq j < J$ overlaid on the bunny manifold.

- Add noise to the signal $\boldsymbol{x}$,

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\varepsilon}\,,$$

  where $\boldsymbol{\varepsilon}(a) \sim \mathcal{N}(0, \sigma^2)$. Set $\sigma = 0.1$. Display $\boldsymbol{W}_J\boldsymbol{y}$ as you displayed $\boldsymbol{W}_J\boldsymbol{x}$, and compare the wavelet coefficients of each signal. In light of Theorem 38 (wavelet inversion), describe in words how you might de-noise this non-smooth signal $\boldsymbol{x}$, keeping in mind that $\boldsymbol{W}_J$ will be invertible even if you only satisfy (2). No need to implement your idea, although you certainly can if you want to!