CMSE 890-001: Spectral Graph Theory and Related Topics, MSU, Spring 2021

Homework 06

Due: March 14, 2021

In this homework you will have some exercises related to graph signal processing. Many of them appeared (or will appear) in our class lectures.

Exercise 1. Let $\boldsymbol{B}$ be a symmetric, $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (here, even though we use λ_k to denote the eigenvalues, $\boldsymbol{B}$ does not have to be a graph Laplacian) and corresponding orthonormal eigenvectors $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n$. Let $\boldsymbol{\Phi}$ be the $n \times n$ matrix whose k^{th} column is $\boldsymbol{\phi}_k$, and let $\boldsymbol{\Lambda}$ be the $n \times n$ diagonal matrix with $\boldsymbol{\Lambda}(k,k) = \lambda_k$. Prove that for any polynomial p,

$$p(\boldsymbol{B}) = \boldsymbol{\Phi} p(\boldsymbol{\Lambda}) \boldsymbol{\Phi}^T \,,$$

where $p(\boldsymbol{C})$ for a matrix $\boldsymbol{C}$ and a polynomial $p(t) = \sum_{j=0}^{m} c_j t^j$ is defined as

$$p(\boldsymbol{C}) := \sum_{j=0}^{m} c_j \boldsymbol{C}^j \,.$$

Exercise 2. Let $G = (V, E)$ be a graph and let $\boldsymbol{B}$ be a matrix such that

$$\forall\, a, b \in V, \;\; a \neq b, \quad b \notin N(a) \implies \boldsymbol{B}(a,b) = 0\,.$$

Prove that

$$\forall\, a, b \in V, \;\; a \neq b, \quad b \notin N_m(a) \implies (\boldsymbol{B}^m)(a,b) = 0\,,$$

where $N_m(a)$ is the m-hop neighborhood of a,

$$N_m(a) := \{ b \in V : b \neq a \text{ and } \operatorname{dist}(a,b) \leq m \}\,,$$

and where $\operatorname{dist}(a,b)$ is the length of the shortest path connecting a to b.

Exercise 3. Let $G = (V, E, w)$ be a connected graph, let $\boldsymbol{x} : V \to \mathbb{R}$ be a graph signal, and let $\boldsymbol{x}_b : V \to \mathbb{R}$ be the graph translation of $\boldsymbol{x}$ to the vertex b, which is defined as:

$$\boldsymbol{x}_b := \sqrt{n} \sum_{k=1}^{n} \widehat{\boldsymbol{x}}(k) \boldsymbol{\psi}_k(b) \boldsymbol{\psi}_k \,.$$

Prove that graph translation preserves the mean of $\boldsymbol{x}$ (analogous to standard translation of Euclidean signals), i.e., show that:

$$\sum_{a \in V} \boldsymbol{x}(a) = \sum_{a \in V} \boldsymbol{x}_b(a)\,.$$

Exercise 4 (30 points)**.** Complete the Python Jupyter notebook `homework06_gsp.ipynb`.