CMSE 890-001: Spectral Graph Theory and Related Topics, MSU, Spring 2021

Lecture 09: Perron-Frobenius Theory

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Lecturer: Matthew Hirn

Our goal in this lecture is to prove Perron-Frobenius Theorem, which we stated in the last lecture for the adjacency matrix $\boldsymbol{M}$ of a graph G, but which in fact has a more general statement for symmetric, non-negative matrices. We will provide the more general statement shortly.

First, let $\boldsymbol{A}$ be an $n \times n$ symmetric matrix with non-negative entries. Define $\text{diag}(\boldsymbol{A})$ as the $n \times n$ matrix with the diagonal of $\boldsymbol{A}$ on its diagonal, and zeroes elsewhere. Note that we can decompose $\boldsymbol{A}$ as:

$$\boldsymbol{A} = \text{diag}(\boldsymbol{A}) + \boldsymbol{M}\,,$$

where $\boldsymbol{M}$ is an $n \times n$, symmetric matrix with zeros down its diagonal. We use the notation $\boldsymbol{M}$ because, indeed, $\boldsymbol{M}$ defines an adjacency matrix of a weighted graph $G(\boldsymbol{A}) = (V, E, w)$ with vertices $V = \{1, \ldots, n\}$, edges

$$E = \{(a, b) \in V \times V : \boldsymbol{M}(a, b) > 0\}\,,$$

and weights $w(a, b) = \boldsymbol{M}(a, b)$. We refer to $G(\boldsymbol{A})$ as the *graph induced by* $\boldsymbol{A}$. We are now ready to state the more general version of the Perron-Frobenius Theorem.

Theorem 26 (Perron-Frobenius Theorem, more general version)**.** *Let $\boldsymbol{A}$ be an $n \times n$ symmetric matrix with non-negative entries, with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, and with corresponding orthonormal eigenvectors $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n$. If the graph $G(\boldsymbol{A})$ induced by $\boldsymbol{A}$ is connected, then*

1. *One can take $\boldsymbol{\phi}_1(a) > 0$ for all $a \in V$*
2. $\mu_1 \geq |\mu_n|$
3. $\mu_1 > \mu_2$

As mentioned last time, we will need two results to prove Theorem 24. The first of these was the following theorem.

Theorem 25. *Let $\boldsymbol{A}$ be a real-valued, symmetric matrix and let μ_1 be the largest eigenvalue of $\boldsymbol{A}$. If*

$$\mu_1 = \frac{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}}\,,$$

for some $\boldsymbol{x} \in \mathbb{R}^n$, then $\boldsymbol{A}\boldsymbol{x} = \mu_1 \boldsymbol{x}$. The same result holds for μ_n, the minimum eigenvalue of $\boldsymbol{A}$.

We proved Theorem 25 in the last lecture. The second result we will need is the following lemma.

Lemma 27. *Let $\boldsymbol{A}$ be an $n \times n$ symmetric matrix with non-negative entries such that the graph $G(\boldsymbol{A})$ induced by $\boldsymbol{A}$ is connected. Let $\boldsymbol{\phi}$ be an eigenvector of $\boldsymbol{A}$ such that $\boldsymbol{\phi}(a) \geq 0$ for all $a \in V$. Then $\boldsymbol{\phi}(a) > 0$ for all $a \in V$.*

Proof. Suppose $\boldsymbol{\phi}$ is not strictly positive. Then there exists some vertex $a \in V$ for which $\boldsymbol{\phi}(a) = 0$. Since $G = G(\boldsymbol{A}) = (V, E, w)$ is connected, there must also be a vertex $b \in V$ (possibly with $b = a$) such that $\boldsymbol{\phi}(b) = 0$ and for some $c \in N(b)$, $\boldsymbol{\phi}(c) > 0$. Let μ be the eigenvalue of $\boldsymbol{\phi}$. Then:

$$0 = \mu\boldsymbol{\phi}(b) = \boldsymbol{A}\boldsymbol{\phi}(b) = \boldsymbol{A}(b,b)\boldsymbol{\phi}(b) + \sum_{v \in N(b)} \boldsymbol{A}(b,v)\boldsymbol{\phi}(v) \geq \boldsymbol{A}(b,c)\boldsymbol{\phi}(c) > 0\,,$$

since $(b,c) \in E$ implies that $\boldsymbol{A}(b,c) > 0$. However, this is a contradiction. □

Now we can prove the Perron-Frobenius Theorem.

Proof of Theorem 24. Let $G = G(\boldsymbol{A}) = (V, E, w)$ be the graph induced by $\boldsymbol{A}$.

First part: Let $\boldsymbol{\phi}_1$ be an eigenvector of $\boldsymbol{A}$ with eigenvalue μ_1 and with unit norm, $\|\boldsymbol{\phi}_1\| = 1$. Set

$$\boldsymbol{y}(a) := |\boldsymbol{\phi}_1(a)|\,, \quad a \in V\,.$$

Notice that $\|\boldsymbol{y}\| = \|\boldsymbol{\phi}_1\| = 1$ as well, so for both vectors their Rayleigh quotient has denominator equal to one. We have:

$$\begin{aligned}
\mu_1 = \boldsymbol{\phi}_1^T\boldsymbol{A}\boldsymbol{\phi}_1 &= \sum_{a \in V} \boldsymbol{A}(a,a)\boldsymbol{\phi}_1(a)^2 + 2\sum_{(a,b) \in E} \boldsymbol{A}(a,b)\boldsymbol{\phi}_1(a)\boldsymbol{\phi}_1(b) \\
&\leq \sum_{a \in V} \boldsymbol{A}(a,a)|\boldsymbol{\phi}_1(a)|^2 + 2\sum_{(a,b) \in E} \boldsymbol{A}(a,b)|\boldsymbol{\phi}_1(a)||\boldsymbol{\phi}_1(b)| \\
&= \sum_{a \in V} \boldsymbol{A}(a,a)\boldsymbol{y}(a)^2 + 2\sum_{(a,b) \in E} \boldsymbol{A}(a,b)\boldsymbol{y}(a)\boldsymbol{y}(b) \\
&= \boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}\,.
\end{aligned}$$

But by the Courant-Fischer Theorem (Theorem 3),

$$\mu_1 = \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \boldsymbol{x} \neq \boldsymbol{0}}} \frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}}\,,$$

which means that we must have $\mu_1 = \boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}$. But then, by Theorem 25, $\boldsymbol{y}$ must be an eigenvector of $\boldsymbol{A}$. Finally, applying Lemma 27, we see that $\boldsymbol{y}$ be must be strictly positive.

Second part: Let $\boldsymbol{\phi}_n$ be an eigenvector of $\boldsymbol{A}$ with eigenvalue μ_n and with unit norm, $\|\boldsymbol{\phi}_n\| = 1$. Similarly to the first part, set

$$\boldsymbol{y}(a) := |\boldsymbol{\phi}_n(a)|\,, \quad a \in V\,.$$

We have:

$$
\begin{aligned}
|\mu_n| = |\boldsymbol{\phi}_n^T \boldsymbol{A} \boldsymbol{\phi}_n| &= \left| \sum_{a \in V} \boldsymbol{A}(a,a) \boldsymbol{\phi}_n(a)^2 + 2 \sum_{(a,b) \in E} \boldsymbol{A}(a,b) \boldsymbol{\phi}_n(a) \boldsymbol{\phi}_n(b) \right| \\
&\leq \sum_{a \in V} \boldsymbol{A}(a,a) |\boldsymbol{\phi}_n(a)|^2 + 2 \sum_{(a,b) \in E} \boldsymbol{A}(a,b) |\boldsymbol{\phi}_n(a)| |\boldsymbol{\phi}_n(b)| \\
&= \sum_{a \in V} \boldsymbol{A}(a,a) \boldsymbol{y}(a)^2 + 2 \sum_{(a,b) \in E} \boldsymbol{A}(a,b) \boldsymbol{y}(a) \boldsymbol{y}(b) \\
&= \boldsymbol{y}^T \boldsymbol{A} \boldsymbol{y} \\
&\leq \mu_1 \,.
\end{aligned}
$$

Third part: Take $\boldsymbol{\phi}_1$ to be the normalized eigenvector of $\boldsymbol{A}$ with eigenvalue μ_1 that is strictly positive (using part 1). Let $\boldsymbol{\phi}_2$ be an eigenvector of $\boldsymbol{A}$ with eigenvalue μ_2, unit norm, $\|\boldsymbol{\phi}_2\| = 1$, and that is orthogonal to $\boldsymbol{\phi}_1$. Following the same method as the previous two parts, set

$$
\boldsymbol{y}(a) := |\boldsymbol{\phi}_2(a)| \,, \quad a \in V \,.
$$

We have that

$$
\begin{aligned}
\mu_2 = \boldsymbol{\phi}_2^T \boldsymbol{A} \boldsymbol{\phi}_2 &= \sum_{a \in V} \boldsymbol{A}(a,a) \boldsymbol{\phi}_2(a)^2 + 2 \sum_{(a,b) \in E} \boldsymbol{A}(a,b) \boldsymbol{\phi}_2(a) \boldsymbol{\phi}_2(b) \\
&\leq \sum_{a \in V} \boldsymbol{A}(a,a) |\boldsymbol{\phi}_2(a)|^2 + 2 \sum_{(a,b) \in E} \boldsymbol{A}(a,b) |\boldsymbol{\phi}_2(a)| |\boldsymbol{\phi}_2(b)| \\
&= \sum_{a \in V} \boldsymbol{A}(a,a) \boldsymbol{y}(a)^2 + 2 \sum_{(a,b) \in E} \boldsymbol{A}(a,b) \boldsymbol{y}(a) \boldsymbol{y}(b) \\
&= \boldsymbol{y}^T \boldsymbol{A} \boldsymbol{y} \\
&\leq \mu_1 \,.
\end{aligned}
$$

Of course we already knew $\mu_1 \geq \mu_2$, but the important parts of the above calculation are the steps in between. Now suppose that $\mu_1 = \mu_2$. Then, by the above calculation, $\boldsymbol{y}^T \boldsymbol{A} \boldsymbol{y} = \mu_1$. Thus, by Theorem 25, $\boldsymbol{y}$ must be an eigenvector of $\boldsymbol{A}$ with eigenvalue μ_1. Since $\boldsymbol{y}$ is a non-negative eigenvector of $\boldsymbol{A}$, we can apply Lemma 27 to conclude that $\boldsymbol{y}$ is strictly positive, which in turn implies that $\boldsymbol{\phi}_2(a) \neq 0$ for all $a \in V$.

Now, since $\langle \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \rangle = 0$ and $\boldsymbol{\phi}_1(a) > 0$ for all $a \in V$, it must be that $\boldsymbol{\phi}_2$ has some positive entries and some negative entries. Since G is connected and since $\boldsymbol{\phi}_2(a) \neq 0$ for all $a \in V$, it must be that there exists an $(u,v) \in E$ such that

$$
\boldsymbol{\phi}_2(u) < 0 < \boldsymbol{\phi}_2(v) \,.
$$

But if that is the case, then

$$\mu_2 = \sum_{a \in V} \boldsymbol{A}(a,a)\boldsymbol{\phi}_2(a)^2 + 2 \sum_{\substack{(a,b)\in E \\ (a,b)\neq(u,v)}} \boldsymbol{A}(a,b)\boldsymbol{\phi}_2(a)\boldsymbol{\phi}_2(b) + \underbrace{2\boldsymbol{A}(u,v)\boldsymbol{\phi}_2(u)\boldsymbol{\phi}_2(v)}_{<0}$$
$$< \sum_{a \in V} \boldsymbol{A}(a,a)|\boldsymbol{\phi}_2(a)|^2 + 2 \sum_{(a,b)\in E} \boldsymbol{A}(a,b)|\boldsymbol{\phi}_2(a)||\boldsymbol{\phi}_2(b)| = \mu_1 \,.$$

Thus $\mu_2 < \mu_1$, which contradicts the assumption that $\mu_2 = \mu_1$; we thus conclude that indeed $\mu_2 < \mu_1$. □

Remark 16. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of the adjacency matrix, $\boldsymbol{M}$, of a graph $G = (V, E)$. One can show if G is connected, then $\mu_1 = -\mu_n$ if and only if G is bipartite. The details can be found in [1, Chapter 4]

We can extend the Perron-Frobenius Theorem to the graph Laplacian (this was already clear) and graph Laplacian like matrices. Here is the result.

Corollary 28. *Let $\widetilde{\boldsymbol{L}}$ be a symmetric, $n \times n$ matrix with non-positive off-diagonal entries, so that*

$$\widetilde{\boldsymbol{L}} = \operatorname{diag}(\widetilde{\boldsymbol{L}}) - \boldsymbol{M} \,,$$

where $\boldsymbol{M}$ is a symmetric, real-valued matrix with non-negative entries and zeroes down its diagonal. Suppose the graph G induced by $\boldsymbol{M}$ is connected. Let $\widetilde{\lambda}_1$ be the smallest eigenvalue of $\widetilde{\boldsymbol{L}}$ with eigenvector $\widetilde{\boldsymbol{\psi}}_1$. Then $\widetilde{\boldsymbol{\psi}}_1$ may be taken to be strictly positive, and $\widetilde{\lambda}_1$ has multiplicity one.

Proof. Set

$$\boldsymbol{A} = \sigma \boldsymbol{I} - \widetilde{\boldsymbol{L}} = (\sigma \boldsymbol{I} - \operatorname{diag}(\widetilde{\boldsymbol{L}})) + \boldsymbol{M} \,.$$

If we set σ such that

$$\sigma \geq \max_{a \in V} \widetilde{\boldsymbol{L}}(a,a) \,,$$

then the matrix $\boldsymbol{A}$ will be a symmetric matrix with non-negative entries such that the graph $G(\boldsymbol{A})$ is connected. We may therefore apply the Perron-Frobenius Theorem (the general version, Theorem 26) to the matrix $\boldsymbol{A}$, and conclude that its largest eigenvalue, μ_1, has multiplicity one, and the eigenvector associated to μ_1, $\boldsymbol{\phi}_1$, may be taken to be strictly positive. But any eigenvector $\boldsymbol{\phi}$ of $\boldsymbol{A}$ is an eigenvector of $\widetilde{\boldsymbol{L}}$, and if $\boldsymbol{A\phi} = \mu\boldsymbol{\phi}$ then $\widetilde{\boldsymbol{L}}\boldsymbol{\phi} = (\sigma - \mu)\boldsymbol{\phi}$, i.e., $\widetilde{\lambda} = \sigma - \mu$ is the corresponding eigenvalue of $\widetilde{\boldsymbol{L}}$. Thus $\widetilde{\lambda}_1 = \sigma - \mu_1$ has multiplicity one. □

References

[1] Daniel A. Spielman. Spectral and algebraic graph theory. Book draft, available at: http://cs-www.cs.yale.edu/homes/spielman/sagt/, 2019.

[2] Michael Perlmutter, Feng Gao, Guy Wolf, and Matthew Hirn. Geometric scattering networks on compact Riemannian manifolds. In *Proceedings of The First Mathematical and Scientific Machine Learning Conference, Proceedings of Machine Learning Research*, volume 107, pages 570–604, 2020.