CMSE 890-001: Spectral Graph Theory and Related Topics, MSU, Spring 2021

Lecture 07: The Cycle Graph and the Path Graph

February 9, 2021

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15 The cycle graph

In order to compute the eigenvectors an eigenvalues of the path graph, P_n, it will be useful, and easier, to compute the eigenvectors and eigenvalues of the cycle graph. Recall the cycle graph on n vertices, which we will denote by $C_n = (V, E)$, is defined as:

$$\begin{aligned} V &= \{1, \ldots, n\}\,, \\ E &= \{(1,2), (2,3), \ldots, (n-1, n), (n, 1)\}\,. \end{aligned}$$

The following theorem shows that the eigenvectors of the cycle graph are the standard, real valued Fourier modes, namely cosine and sine functions at different discrete frequencies.

Theorem 17. *The graph Laplacian of the cycle graph, C_n, has eigenvectors*

$$\begin{aligned} \boldsymbol{x}_k(a) &= \cos(2\pi k a/n)\,, \quad 0 \leq k \leq n/2\,, \\ \boldsymbol{y}_k(a) &= \sin(2\pi k a/n)\,, \quad 1 \leq k \leq \lceil n/2 - 1 \rceil\,. \end{aligned}$$

The eigenvectors $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ have eigenvalue

$$\mu_k = 2 - 2\cos(2\pi k/n)\,.$$

Proof. We prove, via direct calculation, the result for $\boldsymbol{y}_k$. A similar calculation works for $\boldsymbol{x}_k$, which can be found in [1, Chapter 6.5]. We are going to use the following trig identity:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta\,,$$

which implies

$$\sin(\alpha - \beta) + \sin(\alpha + \beta) = 2\sin\alpha\cos\beta\,.$$

Now we make the following calculation for each $a \in V$:

$$\begin{aligned}
\boldsymbol{L}_{C_n}\boldsymbol{y}_k(a) &= 2\boldsymbol{y}_k(a) - \boldsymbol{y}_k(a-1 \mod n) - \boldsymbol{y}_k(a+1 \mod n) \\
&= 2\sin\left(\frac{2\pi ka}{n}\right) - \sin\left(\frac{2\pi k(a-1)}{n}\right) - \sin\left(\frac{2\pi k(a+1)}{n}\right) \\
&= 2\sin\left(\frac{2\pi ka}{n}\right) - \sin\left(\frac{2\pi ka}{n} - \frac{2\pi k}{n}\right) - \sin\left(\frac{2\pi ka}{n} + \frac{2\pi k}{n}\right) \\
&= 2\sin\left(\frac{2\pi ka}{n}\right) - 2\sin\left(\frac{2\pi ka}{n}\right)\cos\left(\frac{2\pi k}{n}\right) \\
&= \left(2 - 2\cos\left(\frac{2\pi k}{n}\right)\right)\sin\left(\frac{2\pi ka}{n}\right) \\
&= (2 - 2\cos(2\pi k/n))\boldsymbol{y}_k(a)\,.
\end{aligned}$$

□

Remark 12. If we order the eigenvalues of $\boldsymbol{L}_{C_n}$ in increasing order, $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ we have:

$$\begin{aligned}
\lambda_1 &= 0 \quad \text{with} \quad \boldsymbol{\psi}_1(a) = \boldsymbol{x}_0(a) = \mathbf{1}(a) \\
\lambda_2 &= 2 - 2\cos(2\pi/n) \quad \text{with} \quad \boldsymbol{\psi}_2(a) = \boldsymbol{x}_1(a) = \cos(2\pi a/n) \\
\lambda_3 &= 2 - 2\cos(2\pi/n) \quad \text{with} \quad \boldsymbol{\psi}_3(a) = \boldsymbol{y}_1(a) = \sin(2\pi a/n)\,,
\end{aligned}$$

and $\lambda_4 > \lambda_3$. Thus the eigenvector embedding of the cycle graph is

$$a \mapsto (\boldsymbol{\psi}_2(a), \boldsymbol{\psi}_3(a)) = (\cos(2\pi a/n), \sin(2\pi a/n))\,.$$

This is a really good embedding, as it gives evenly spaced samples on the unit circle! See also Exercise 5 from Homework 02.

16 The path graph

Now that we have computed the eigenvalues and eigenvectors of the cycle graph, C_n, we are in a good position to compute the eigenvalues and eigenvectors of the path graph, P_n.

Theorem 18. *The graph Laplacian of the path graph, P_n, has eigenvalues*

$$\lambda_{k+1} = 2 - 2\cos(\pi k/n)\,, \quad 0 \leq k < n\,,$$

with eigenvectors

$$\boldsymbol{\psi}_{k+1}(a) = \cos\left(\frac{\pi ka}{n} - \frac{\pi k}{2n}\right)\,, \quad 0 \leq k < n\,.$$

Proof. Notice the values λ_{k+1} are the eigenvalues of C_{2n}, the cycle graph on $2n$ vertices. This is not an accident and we will use C_{2n} to prove path graph result. The idea is to view P_n as the quotient of C_{2n}. In particular:

$$P_n = C_{2n}/\sim,$$

where $\sim$ identifies $a \in V(C_{2n})$ with $2n+1-a \in V(C_{2n})$, yielding the vertex $a \in V(P_n)$. The edges $(1, 2n), (n, n+1) \in E(C_{2n})$ are eliminated under this equivalence relation, and the edges $(a, a+1), (2n-a, 2n+1-a) \in E(C_{2n})$ are identified as the same edge for $1 \leq a < n$, yielding the edge $(a, a+1) \in E(P_n)$. See Figure 18 for a picture illustrating this equivalence relation.

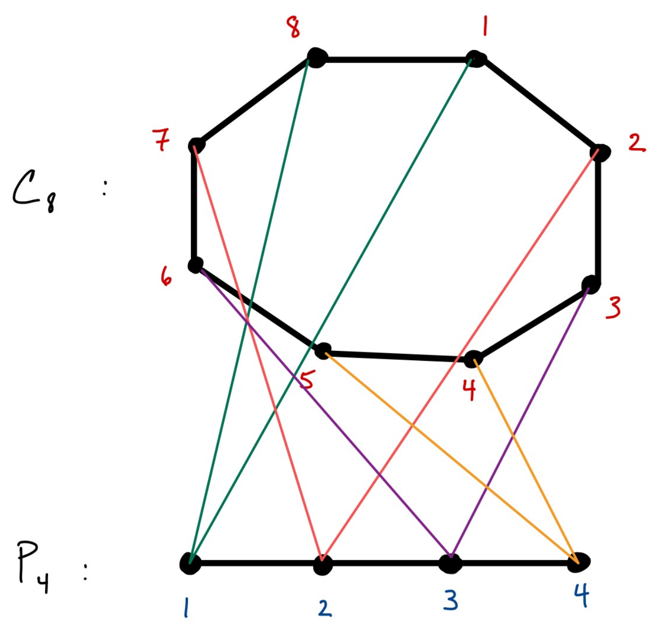

Figure 18: Illustration of the equivalence relation $P_n = C_{2n}/\sim$ for $n = 4$. Pairs of lines of the same color identify two vertices of C_{2n} with one vertex in P_n.

Now consider the function $\boldsymbol{\phi}_{k+1} : V(C_{2n}) \to \mathbb{R}$ defined as

$$\boldsymbol{\phi}_{k+1}(a) := \cos\left(\frac{\pi k a}{n} - \frac{\pi k}{2n}\right), \quad 1 \leq a \leq 2n.$$

Notice that $\boldsymbol{\phi}_{k+1}(a) = \boldsymbol{\psi}_{k+1}(a)$ for $1 \leq a \leq n$. In fact, though, $\boldsymbol{\phi}_{k+1}$ is well defined on P_n under the equivalence relation $\sim$ (not just by restricting to $1 \leq a \leq n$). In other words,

$$\boldsymbol{\phi}_{k+1}(a) = \boldsymbol{\phi}_{k+1}(2n+1-a), \quad \forall\, 1 \leq a \leq n. \tag{20}$$

Indeed,

$$\begin{aligned}
\boldsymbol{\phi}_{k+1}(2n+1-a) &= \cos\left(\frac{\pi k(2n+1-a)}{n} - \frac{\pi k}{2n}\right) \\
&= \cos\left(2\pi k + \frac{\pi k}{n} - \frac{\pi k a}{n} - \frac{\pi k}{2n}\right) \\
&= \cos\left(-\frac{\pi k a}{n} + \frac{\pi k}{2n}\right) \\
&= \cos\left(\frac{\pi k a}{n} - \frac{\pi k}{2n}\right) \\
&= \boldsymbol{\phi}_{k+1}(a)\,.
\end{aligned}$$

Now we will use the trig identity

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin\alpha \sin\beta\,,$$

to write $\boldsymbol{\phi}_{k+1}$ in terms of the eigenvectors $\boldsymbol{x}_k$, $\boldsymbol{y}_k$ of $\boldsymbol{L}_{C_{2n}}$, which we computed in Theorem 17. We have

$$\begin{aligned}
\boldsymbol{\phi}_{k+1}(a) &= \cos\left(\frac{2\pi k a}{2n} - \frac{\pi k}{2n}\right) \\
&= \cos\left(\frac{2\pi k a}{2n}\right)\cos\left(\frac{\pi k}{2n}\right) + \sin\left(\frac{2\pi k a}{2n}\right)\sin\left(\frac{\pi k}{2n}\right) \\
&= \cos\left(\frac{\pi k}{2n}\right)\boldsymbol{x}_k(a) + \sin\left(\frac{\pi k}{2n}\right)\boldsymbol{y}_k(a)\,.
\end{aligned}$$

Thus $\boldsymbol{\phi}_{k+1}$ is a linear combination of $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$. Since $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ have the same eigenvalue λ_{k+1}, this means that $\boldsymbol{\phi}_{k+1}$ is an eigenvector of $\boldsymbol{L}_{C_{2n}}$ with eigenvalue λ_{k+1} as well, i.e.,

$$\boldsymbol{L}_{C_{2n}}\boldsymbol{\phi}_{k+1} = \lambda_{k+1}\boldsymbol{\phi}_{k+1}\,. \tag{21}$$

We can now show that $\boldsymbol{\psi}_{k+1}$ is an eigenvector of $\boldsymbol{L}_{P_n}$ with eigenvalue λ_{k+1}. Using (20) and (21), we make the following calculation:

$$\begin{aligned}
\forall\, 1 < a < n\,, \quad \boldsymbol{L}_{P_n}\boldsymbol{\psi}_{k+1}(a) &= 2\boldsymbol{\psi}_{k+1}(a) - \boldsymbol{\psi}_{k+1}(a-1) - \boldsymbol{\psi}_{k+1}(a+1) \\
&= \frac{1}{2}\Big(2\boldsymbol{\phi}_{k+1}(a) - \boldsymbol{\phi}_{k+1}(a-1) - \boldsymbol{\phi}_{k+1}(a+1) + \ldots \\
&\ldots + 2\boldsymbol{\phi}_{k+1}(2n+1-a) - \boldsymbol{\phi}_{k+1}(2n+1-(a-1)) - \boldsymbol{\phi}_{k+1}(2n+1-(a+1))\Big) \\
&= \frac{1}{2}\Big(\boldsymbol{L}_{C_{2n}}\boldsymbol{\phi}_{k+1}(a) + \boldsymbol{L}_{C_{2n}}\boldsymbol{\phi}_{k+1}(2n+1-a)\Big) \\
&= \frac{1}{2}\Big(\lambda_{k+1}\boldsymbol{\phi}_{k+1}(a) + \lambda_{k+1}\boldsymbol{\phi}_{k+1}(2n+1-a)\Big) \\
&= \lambda_{k+1}\boldsymbol{\psi}_{k+1}(a)\,.
\end{aligned}$$

For $a = 1$ we have:

$$\begin{aligned}\boldsymbol{L}_{P_n}\boldsymbol{\psi}_{k+1}(1) = \boldsymbol{\psi}_{k+1}(1) - \boldsymbol{\psi}_{k+1}(2) &= 2\boldsymbol{\psi}_{k+1}(1) - \boldsymbol{\psi}_{k+1}(2) - \boldsymbol{\psi}_{k+1}(1) \\ &= 2\boldsymbol{\phi}_{k+1}(1) - \boldsymbol{\phi}_{k+1}(2) - \boldsymbol{\phi}_{k+1}(2n) \\ &= \boldsymbol{L}_{C_{2n}}\boldsymbol{\phi}_{k+1}(1) = \lambda_{k+1}\boldsymbol{\phi}_{k+1}(1) = \lambda_{k+1}\boldsymbol{\psi}_{k+1}(1)\,.\end{aligned}$$

A similar calculation works for $a = n$ as well. □

Remark 13. In Sections 12 and 14 we estimated the second eigenvalue of $\boldsymbol{L}_{P_n}$ as

$$\frac{6}{(n-1)(n+1)} \leq \lambda_2(P_n) \leq \frac{12}{n(n+1)}\,.$$

Theorem 18 proves this is a very good estimate. Indeed, recall they Taylor series of $\cos(u)$ is

$$\cos(u) = 1 - \frac{u^2}{2} + O(u^4)\,.$$

Thus,

$$\lambda_2(P_n) = 2(1 - \cos(\pi/n)) = 2\left(1 - 1 + \frac{1}{2}\frac{\pi^2}{n^2} + O(n^{-4})\right) = \frac{\pi^2}{n^2} + O(n^{-4})\,.$$

Remark 14. Recall in Figure 9 and Figure 10 we made the empirical observation that the eigenvectors of the path graph on 10 vertices increase in frequency with increasing eigenvalue. This theorem proves that observation was not an accident, and that it holds for any path graph P_n. Indeed, since

$$\boldsymbol{\psi}_k(a) = \cos\left(\frac{\pi(k-1)a}{n} - \frac{\pi(k-1)}{2n}\right)\,, \quad 1 \leq k \leq n\,,$$

we see the frequency of $\boldsymbol{\psi}_k$ is $\pi(k-1)/n$, which increases as k increases. Using this formula for $\boldsymbol{\psi}_k$ one can show that if there is no vertex $a \in V_{P_n}$ for which $\boldsymbol{\psi}_k(a) = 0$, then there are exactly $k-1$ edges $(a,b) \in E_{P_n}$ for which $\boldsymbol{\psi}_k(a)\boldsymbol{\psi}_k(b) < 0$, which corresponds to when $\boldsymbol{\psi}_k$ changes sign over an edge. In fact, we will be able to generalize this result to weighted path graphs, and we will even be able to formulate an analogous result for general graphs. These results will clarify what we mean by the frequency of an eigenvector of the graph Laplacian for a general graph, and they will show that the frequency of these eigenvectors increases with increasing eigenvalue. We will work towards this result over the next few lectures, along the way covering items of independent interest as well.

References

[1] Daniel A. Spielman. Spectral and algebraic graph theory. Book draft, available at: http://cs-www.cs.yale.edu/homes/spielman/sagt/, 2019.

[2] Michael Perlmutter, Feng Gao, Guy Wolf, and Matthew Hirn. Geometric scattering networks on compact Riemannian manifolds. In *Proceedings of The First Mathematical and Scientific Machine Learning Conference, Proceedings of Machine Learning Research*, volume 107, pages 570–604, 2020.