CMSE 890-001: Spectral Graph Theory and Related Topics, MSU, Spring 2021
Lecture 05: Product Graphs and Star Graphs
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10 Product graphs

Let $G = (V, E, v)$ and $H = (W, F, w)$ be two weighted graphs. Define the *product graph* $G \times H$ as the graph with vertex set $V \times W$ and with edge set $E_{G \times H}$:

- $((a_1, b), (a_2, b))$ with weight $w_{G\times H}((a_1, b), (a_2, b)) = v(a_1, a_2)$ where $(a_1, a_2) \in E$; and
- $((a, b_1), (a, b_2))$ with weight $w_{G\times H}((a, b_1), (a, b_2)) = w(b_1, b_2)$ where $(b_1, b_2) \in F$.

Let P_n be the path graph on n vertices. The graph $P_m \times P_n$ is the $m \times n$ *grid graph*; see Figure 15 for a picture.

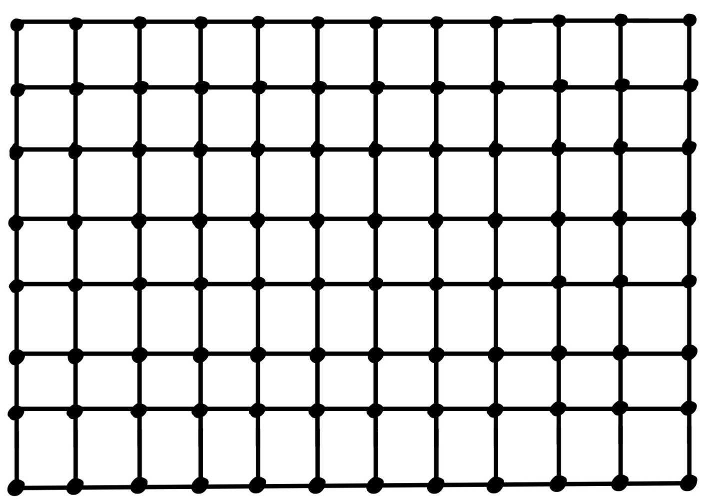

Figure 15: The 8×12 grid graph.

Let's draw the grid graph in $\mathbb{R}^2$ using the eigenvector embedding of Section 9; the embedding of the 8×12 grid graph is given in Figure 16. Comparing to Figure 15, the eigenvector embedding is a remarkably good drawing of the grid graph given that it used nothing specific to the grid graph. The reason for this is that the eigenvectors of a product graph $G \times H$ are the product of the eigenvectors of the two graphs G and H. The next theorem explains.

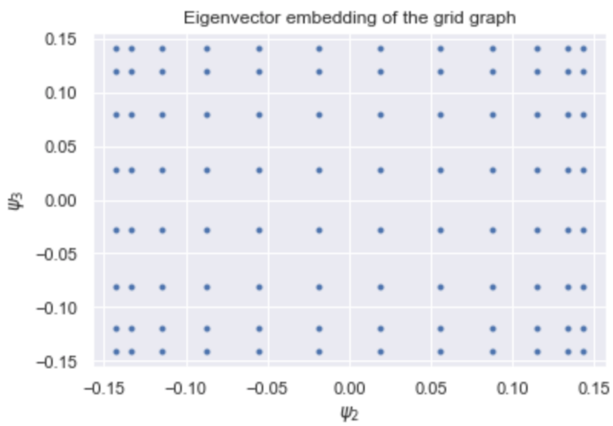

Figure 16: The eigenvector embedding of the 8×12 grid graph.

Theorem 9. *Let $G = (V, E, v)$ and $H = (W, F, w)$ be weighted graphs with graph Laplacian eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$, and graph Laplacian eigenvectors $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$ and $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m$, respectively. Then $\boldsymbol{L}_{G\times H}$ has eigenvalues*

$$\{\lambda_i + \mu_j : 1 \leq i \leq n,\ \ 1 \leq j \leq m\}\,,$$

with eigenvectors

$$\boldsymbol{\psi}_{i,j}(a, b) = \boldsymbol{\alpha}_i(a)\boldsymbol{\beta}_j(b)\,, \quad \forall\, 1 \leq i \leq n,\ \ 1 \leq j \leq m\,.$$

Proof. We drop the (i, j) sub-indices but otherwise everything is defined as in the statement of theorem. Recall from (7) that

$$\boldsymbol{L}\boldsymbol{x}(c) = \sum_{d\in N(c)} \widetilde{w}(c, d)(\boldsymbol{x}(c) - \boldsymbol{x}(d))\,,$$

for any graph $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{w})$. Let us apply this fact and make the following calculation:

$$\begin{aligned}
\boldsymbol{L}_{G\times H}\boldsymbol{\psi}(a, b) &= \sum_{(c,d)\in N(a,b)} w_{G\times H}((a, b), (c, d))(\boldsymbol{\psi}(a, b) - \boldsymbol{\psi}(c, d)) \\
&= \sum_{(a,a_1)\in E} v(a, a_1)(\boldsymbol{\psi}(a, b) - \boldsymbol{\psi}(a_1, b)) + \sum_{(b,b_1)\in F} w(b, b_1)(\boldsymbol{\psi}(a, b) - \boldsymbol{\psi}(a, b_1)) \\
&= \sum_{(a,a_1)\in E} v(a, a_1)(\boldsymbol{\alpha}(a)\boldsymbol{\beta}(b) - \boldsymbol{\alpha}(a_1)\boldsymbol{\beta}(b)) + \sum_{(b,b_1)\in F} w(b, b_1)(\boldsymbol{\alpha}(a)\boldsymbol{\beta}(b) - \boldsymbol{\alpha}(a)\boldsymbol{\beta}(b_1)) \\
&= \boldsymbol{\beta}(b) \sum_{a_1\in N(a)} v(a, a_1)(\boldsymbol{\alpha}(a) - \boldsymbol{\alpha}(a_1)) + \boldsymbol{\alpha}(a) \sum_{b_1\in N(b)} w(b, b_1)(\boldsymbol{\beta}(b) - \boldsymbol{\beta}(b_1)) \\
&= \boldsymbol{\beta}(b)\boldsymbol{L}_G\boldsymbol{\alpha}(a) + \boldsymbol{\alpha}(a)\boldsymbol{L}_H\boldsymbol{\beta}(b) \\
&= \boldsymbol{\beta}(b)\lambda\boldsymbol{\alpha}(a) + \boldsymbol{\alpha}(a)\mu\boldsymbol{\beta}(b) \\
&= (\lambda + \mu)\boldsymbol{\alpha}(a)\boldsymbol{\beta}(b) = (\lambda + \mu)\boldsymbol{\psi}(a, b)\,.
\end{aligned}$$

□

11 The star graph

In your first homework you were asked to make a conjecture regarding the eigenvalues of the star graph on n vertices. Let us now prove that they are 0, 1 (with multiplicity $n-2$), and n. Recall the star graph on n vertices, which we will denote by $S_n = (V, E)$, is defined as

$$\begin{aligned} V &= \{1, \ldots, n\}\,, \\ E &= \{(1, a) : 2 \leq a \leq n\}\,. \end{aligned}$$

We will first need the following lemma, which is of interest even for general graphs.

Lemma 10. *Let $G = (V, E)$ be a graph and let $a, b \in V$ be vertices of degree one that are both connected to another vertex $c \in V$. Then, the vector $\boldsymbol{\psi} = \boldsymbol{\delta}_a - \boldsymbol{\delta}_b$ is an eigenvector of $\boldsymbol{L}$ with eigenvalue* 1.

Proof. We will use (7) and calculate $\boldsymbol{L}\boldsymbol{\psi}(v)$ for each vertex $v \in V$. Applying (7) we have:

$$\boldsymbol{L}\boldsymbol{\psi}(v) = \sum_{u \in N(v)} (\boldsymbol{\psi}(v) - \boldsymbol{\psi}(u)) = \sum_{u \in N(v)} (\boldsymbol{\delta}_a(v) - \boldsymbol{\delta}_b(v) - \boldsymbol{\delta}_a(u) + \boldsymbol{\delta}_b(u))\,. \tag{13}$$

Now we have four cases.

1. $v = a$. Then c is only the neighbor of $v = a$ and (13) is equal to $(\boldsymbol{\delta}_a(a) - \boldsymbol{\delta}_b(a) - \boldsymbol{\delta}_a(c) + \boldsymbol{\delta}_b(c)) = 1 = \boldsymbol{\psi}(a)$.

2. $v = b$. Again c is the only neighbor of $v = b$ and (13) is equal to $(\boldsymbol{\delta}_a(b) - \boldsymbol{\delta}_b(b) - \boldsymbol{\delta}_a(c) + \boldsymbol{\delta}_b(c)) = -1 = \boldsymbol{\psi}(b)$.

3. $v = c$. In this case a and b are neighbors of $v = c$, and c may have other neighbors too. We write (13) as:

$$\begin{aligned} (13) = {} & (\boldsymbol{\delta}_a(c) - \boldsymbol{\delta}_b(c) - \boldsymbol{\delta}_a(a) + \boldsymbol{\delta}_b(a)) \\ & + (\boldsymbol{\delta}_a(c) - \boldsymbol{\delta}_b(c) - \boldsymbol{\delta}_a(b) + \boldsymbol{\delta}_b(b)) \\ & + \sum_{\substack{u \in N(c) \\ u \neq a, b}} (\boldsymbol{\delta}_a(c) - \boldsymbol{\delta}_b(c) - \boldsymbol{\delta}_a(u) + \boldsymbol{\delta}_b(u)) \\ = {} & -1 + 1 + 0 = 0 = \boldsymbol{\psi}(c)\,. \end{aligned}$$

4. $v \neq a, b, c$. In this case we may write (13) as

$$(13) = \sum_{\substack{u \in N(v) \\ u \neq a, b}} (\boldsymbol{\delta}_a(v) - \boldsymbol{\delta}_b(v) - \boldsymbol{\delta}_a(u) + \boldsymbol{\delta}_b(u)) = 0 = \boldsymbol{\psi}(v)\,.$$

□

As a corollary we have the following lemma which will also be useful.

Lemma 11. *Let $G = (V, E)$ be a graph, let $a, b \in V$ be vertices of degree one that are both connected to another vertex $c \in V$, and let $\boldsymbol{\phi}$ be an eigenvector of $\boldsymbol{L}$ with eigenvalue $\lambda \neq 1$. Then $\boldsymbol{\phi}(a) = \boldsymbol{\phi}(b)$.*

Proof. By Lemma 10, $\boldsymbol{\psi} = \boldsymbol{\delta}_a - \boldsymbol{\delta}_b$ is an eigenvector of $\boldsymbol{L}$ with eigenvalue 1. Furthermore, since $\boldsymbol{\phi}$ is also an eigenvector of $\boldsymbol{L}$ but with eigenvalue $\lambda \neq 1$, we know by Exercise 1 of Homework 01 that $\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle = 0$. Therefore:

$$0 = \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \sum_{v \in V} \boldsymbol{\phi}(v)\boldsymbol{\psi}(v) = \sum_{v \in V} \boldsymbol{\phi}(v)(\boldsymbol{\delta}_a(v) - \boldsymbol{\delta}_b(v)) = \boldsymbol{\phi}(a) - \boldsymbol{\phi}(b)\,.$$

□

Now we can prove the following theorem about the star graph.

Theorem 12. *The star graph S_n has eigenvalue 0 with multiplicity 1, eigenvalue 1 with multiplicity $n-2$, and eigenvalue n with multiplicity 1.*

Proof. Since the star graph is connected we know it has eigenvalue 0 with multiplicity 1 and the eigenvector is $\mathbf{1}$. Notice that a and $a+1$, for $2 \leq a \leq n-1$, are vertices of S_n of degree 1 both connected to vertex $c = 1$. Therefore

$$\boldsymbol{\psi}_a = \boldsymbol{\delta}_a - \boldsymbol{\delta}_{a+1}\,, \quad \forall\, 2 \leq a \leq n-1\,,$$

are eigenvectors of $\boldsymbol{L}$, each with eigenvalue 1. Even though $\{\boldsymbol{\psi}_a : 2 \leq a \leq n-1\}$ are not orthogonal, they are independent, and so the eigenvalue 1 must have multiplicity at least $n-2$. Thus we just need to determine λ_n.

In your current homework (Exercise 2 of Homework 02), you are asked to show that the trace of a symmetric, real-valued $n \times n$ matrix is equal to the sum of its eigenvalue. Let us apply this to $\boldsymbol{L}$:

$$n - 2 + \lambda_n = \sum_{i=1}^{n} \lambda_i = \mathrm{Tr}(\boldsymbol{L}) = \mathrm{Tr}(\boldsymbol{D} - \boldsymbol{M}) = \mathrm{Tr}(\boldsymbol{D}) = \sum_{a \in V} \deg(a)\,. \tag{14}$$

In S_n we have one vertex of degree $n-1$ and $n-1$ vertices of degree 1. Therefore (14) reads:

$$n - 2 + \lambda_n = 2n - 2 \quad \Longrightarrow \quad \lambda_n = n\,.$$

That completes the theorem, but as a bonus we can we can compute the eigenvector associated to λ_n as well. By Lemma 11 we know that $\boldsymbol{\psi}_n(a) = \boldsymbol{\psi}(a+1)$ for all $2 \leq a \leq n-1$ which means that $\boldsymbol{\psi}_n(a)$ is constant over $2 \leq a \leq n$ (the points of the star). Let us set $\boldsymbol{\psi}_n(a) = 1$ for $2 \leq a \leq n$. To determine $\boldsymbol{\psi}_n(1)$, we note that on the other hand, $\boldsymbol{\psi}_n$ must also be orthogonal $\mathbf{1}$. Therefore:

$$0 = \langle \boldsymbol{\psi}_n, \mathbf{1} \rangle = \sum_{a \in V} \boldsymbol{\psi}_n(a) = n - 1 + \boldsymbol{\psi}_n(1)\,,$$

and we see that $\boldsymbol{\psi}_n(1) = -(n-1)$. □

References

[1] Daniel A. Spielman. Spectral and algebraic graph theory. Book draft, available at: http://cs-www.cs.yale.edu/homes/spielman/sagt/, 2019.

[2] Michael Perlmutter, Feng Gao, Guy Wolf, and Matthew Hirn. Geometric scattering networks on compact Riemannian manifolds. In *Proceedings of The First Mathematical and Scientific Machine Learning Conference, Proceedings of Machine Learning Research*, volume 107, pages 570–604, 2020.