CMSE 890-001: Spectral Graph Theory and Related Topics, MSU, Spring 2021

Homework 04

Due: February 19, 2021

This first exercise is a linear algebra problem that shows that if I write the vertices of a graph down in one order, and you write the vertices of a graph down in another order, then the eigenvalues of any matrix associated to that graph will not change and the entries of the eigenvectors will permute according to the permutation that relates my way of writing down the graph to your way of writing down the graph.

Exercise 1. Let $\mathbf{\Pi}$ be an $n \times n$ permutation matrix. That is, there is a permutation $\pi : V \to V$ such that

$$\mathbf{\Pi}(a, b) = \begin{cases} 1 & a = \pi(b) \\ 0 & a \neq \pi(b) \end{cases} .$$

Let $\boldsymbol{A}$ be an $n \times n$ real-valued, symmetric matrix. Prove that if

$$\boldsymbol{A}\boldsymbol{\phi} = \mu\boldsymbol{\phi} \, ,$$

then

$$(\mathbf{\Pi}\boldsymbol{A}\mathbf{\Pi}^T)\mathbf{\Pi}\boldsymbol{\phi} = \mu(\mathbf{\Pi}\boldsymbol{\phi}) \, .$$

Now let μ_1 be the largest eigenvalue of the adjacency matrix, $\boldsymbol{M}_G$, of a graph $G = (V, E)$. In Lecture 8 we proved that $\mu_1(G) \geq d_{\text{avg}}(G(S))$ for all $S \subseteq V$ using the Cauchy Interlacing Theorem. However, this was a bit overkill. The next exercise asks you to prove the same result using the test vector technique.

Exercise 2. Let $G = (V, E)$, let $S \subseteq V$, and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of $\boldsymbol{M}_G$. Come up with an appropriate test vector, $\boldsymbol{x}_S$, and prove that

$$\mu_1 \geq \frac{\boldsymbol{x}_S^T \boldsymbol{M} \boldsymbol{x}_S}{\boldsymbol{x}_S^T \boldsymbol{x}_S} = d_{\text{avg}}(G(S)) \, .$$

We observed in class that the Cauchy Interlacing Theorem cannot be applied to the graph Laplacian since removing a vertex from G will change the degree of the vertices previously connected to it. However, there are interlacing results for the graph Laplacian in certain scenarios. The next exercise asks you to prove one such result.

Exercise 3. Let $G = (V, E)$ be a non-complete graph and let H be the graph obtained by adding a single edge to G. Prove

$$\lambda_2(G) \leq \lambda_2(H) \leq \lambda_2(G) + 2 \, .$$

Come up with a simple example to show that the upper bound on $\lambda_2(H)$ cannot be improved. That is, come up with a graph G such that $\lambda_2(H) = \lambda_2(G) + 2$.

We saw in Homework 2 that the cycle graph has a very nice embedding as a discretely sampled circle, and then we verified this result theoretically in Lecture 7. In the next exercise, we'll numerically consider a weighted cycle graph and we'll see how the embedding changes.

Exercise 4. Let $C_{2n} = (V, E, w)$ be a weighted cycle graph with weights that linearly increase and then linearly decrease over the cycle:

$$w(a, a+1 \text{ mod } 2n) = \begin{cases} a & 1 \leq a \leq n \\ 2n+1-a & n-1 \leq a \leq 2n \end{cases} .$$

Compute the eigenvector embedding $a \mapsto (\boldsymbol{\psi}_2(a), \boldsymbol{\psi}_3(b))$ of this weighted cycle graph. Explain the result and compare to the embedding of the unweighted cycle graph. Turn in a Python Jupyter notebook showing your work and that I can run.