CMSE 890-001: Spectral Graph Theory and Related Topics, MSU, Spring 2021

Homework 02

Due: February 5, 2021

Like in the first homework, these first two exercises are linear algebra problems meant to help you get comfortable working with eigenvectors and eigenvalues of matrices.

Exercise 1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices. We say that $\boldsymbol{A}$ is *similar* to $\boldsymbol{B}$ is there exists an $n \times n$ invertible matrix $\boldsymbol{Q}$ such that

$$\boldsymbol{B} = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}\,.$$

Prove that similar matrices have the same eigenvalues.

Exercise 2. Recall that for an $n \times n$ matrix $\boldsymbol{C}$ the *trace* of $\boldsymbol{C}$ is the sum of its diagonal entries,

$$\mathrm{Tr}(\boldsymbol{C}) := \sum_{i=1}^{n} \boldsymbol{C}(i,i)\,.$$

Let $\boldsymbol{A}$ be an $m \times n$ matrix and let $\boldsymbol{B}$ be an $n \times m$ matrix. Prove:

$$\mathrm{Tr}(\boldsymbol{A}\boldsymbol{B}) = \mathrm{Tr}(\boldsymbol{B}\boldsymbol{A})\,. \tag{1}$$

Use (1) and Exercise 2 from Homework 01 to prove that for any $n \times n$, real-valued, symmetric matrix $\boldsymbol{C}$,

$$\mathrm{Tr}(\boldsymbol{C}) = \sum_{i=1}^{n} \mu_i\,,$$

where $\mu_1, \ldots, \mu_n$ are the eigenvalues of $\boldsymbol{C}$.

In class we proved that $\lambda_2 = n$ for K_n, the complete graph on n vertices, and said that this was a "large" value for λ_2, which indicated that K_n is well connected. Let us show, in fact, that n is the maximum value for λ_2.

Exercise 3. Let $G = (V, E)$ be a graph, let $\boldsymbol{L}$ be its graph Laplacian, and let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\boldsymbol{L}$. Prove

$$\mathrm{Tr}(\boldsymbol{L}) = \sum_{a \in V} \deg(a) \leq (n-1)n\,. \tag{2}$$

Now use Exercise 2 and (2) to prove

$$\lambda_2 \leq n\,.$$

Now we strengthen Theorem 5, which showed that $\lambda_2 > 0$ if and only if G is connected.

Exercise 4. Prove the following generalization of Theorem 5. Let $G = (V, E, w)$ be a weighted graph and let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of its graph Laplacian $\boldsymbol{L}$. Then G has exactly k connected components if and only if $\lambda_i = 0$ for all $1 \leq i \leq k$ and $\lambda_{k+1} > 0$.

Finally, let's do some programming for drawing graphs with eigenvectors.

Exercise 5. Write code to compute the two-dimensional graph Laplacian eigenvector embedding

$$\forall\, a \in V\,, \quad a \mapsto (\boldsymbol{\psi}_2(a), \boldsymbol{\psi}_3(a)) \in \mathbb{R}^2\,,$$

of a graph G. Test your code on the cycle graph with $n = 20$ vertices (you will need to write a function to generate its adjacency matrix) and the star graph with $n = 20$ vertices (you can use your function from the first homework to generate the adjacency matrix for the star graph). Most likely your star graph embedding will look pretty bad - explain why (later on we will see why the cycle graph looks good). Turn in a Python Jupyter notebook with your work (plus your function file if you load your functions separately).