CMSE 890-001: Spectral Graph Theory and Related Topics, MSU, Spring 2021

Homework 01

Due: January 29, 2021

These first two exercises are linear algebra problems meant to help you get comfortable working with eigenvectors and eigenvalues of matrices.

Exercise 1. Orthogonal eigenvectors. Let $\boldsymbol{M}$ be a symmetric, real valued matrix, and let $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ be two eigenvectors of $\boldsymbol{M}$ with eigenvalues μ and ν, respectively, i.e.,

$$\boldsymbol{M}\boldsymbol{\psi} = \mu\boldsymbol{\psi} \quad \text{and} \quad \boldsymbol{M}\boldsymbol{\phi} = \nu\boldsymbol{\phi}\,.$$

Prove that if $\mu \neq \nu$, then $\boldsymbol{\psi}$ must be orthogonal to $\boldsymbol{\phi}$. Note that your proof should exploit the symmetry of $\boldsymbol{M}$, as this statement is false otherwise.

Exercise 2. Spectral decomposition. Let $\boldsymbol{M}$ be a symmetric, real valued matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and orthonormal eigenvectors $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n$. Let $\boldsymbol{\Psi}$ be the orthogonal matrix whose i^{th} column is $\boldsymbol{\psi}_i$. Prove that

$$\boldsymbol{\Psi}^T \boldsymbol{M} \boldsymbol{\Psi} = \boldsymbol{\Lambda}\,,$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on its diagonal. Conclude that

$$\boldsymbol{M} = \boldsymbol{\Psi}\boldsymbol{\Lambda}\boldsymbol{\Psi}^T = \sum_{i=1}^{n} \lambda_i \boldsymbol{\psi}_i \boldsymbol{\psi}_i^T\,.$$

Now let's prove something about the graph Laplacian quadratic form that we stated in lecture 02, but did not explain. We will use this fact repeatedly.

Exercise 3. Graph Laplacian quadratic form. Let $G = (V, E, w)$ be a weighted graph and let $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{M}$ be its graph Laplacian. Prove that

$$\boldsymbol{x}^T \boldsymbol{L} \boldsymbol{x} = \sum_{(a,b)\in E} w(a,b)(\boldsymbol{x}(a) - \boldsymbol{x}(b))^2\,.$$

Now let's do some programming work.

Exercise 4. Path graph frequency. In the `path_graph_frequency` notebook (posted online), the `spectral_graph_theory_fcns` file is missing. In order to run the notebook, you will need to fill in the missing functions. In particular, for cell 02 you need to write a function `path_graph(n)` that returns the adjacency matrix of the path graph on n vertices. Additionally, for cell 03 write the function `graph_laplacian(M)` that takes as input an adjacency matrix $\boldsymbol{M}$ and returns the graph Laplacian $\boldsymbol{L}$. Turn in these two functions plus other support functions you wrote (if any).

Exercise 5. Star graph. Recall the star graph on n vertices is the graph $G = (V, E)$ with

$$
\begin{aligned}
V &= \{1, \ldots, n\}\,, \\
E &= \{(1,2), (1,3), \ldots, (1,n)\}\,.
\end{aligned}
$$

Write a function, `star_graph`, that returns the adjacency matrix $\boldsymbol{M}$ of the star graph. Then, using your function `graph_laplacian` from the previous exercise, compute the graph Laplacian $\boldsymbol{L}$ of the star graph and compute its eigenvalues. Do this for several different numbers of vertices n and make a conjecture as to what the eigenvalues of the star graph on n vertices are (no need to prove your conjecture, that will come later). Turn in a Python Jupyter notebook that shows your work and that I can run.