Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 17: Estimating Pointwise Regularity

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A wavelet transform, even with $\psi = (-1)^n \theta^{(n)}$ for θ a Gaussian, may have a maxima line that converges to a point v even though f is regular at v (i.e., f is Lipschitz α at v for $\alpha > 1$); see Figure 23, and the maxima line that converges to $v = 0.23$. To distinguish such points from singular points it is necessary to measure the decay of the modulus maxima amplitude.

To interpret more easily the pointwise conditions (38) and (39) of Theorem 5.5, suppose that for $s < s_0$ all modulus maxima that converge to v are included in a cone $\mathcal{C}_v$ defined as:

$$\mathcal{C}_v = \{(u, s) \in \mathbb{R} \times (0, \infty) : |u - v| \leq Cs\}$$

Figure 25 gives an illustration. In general this will not be true, in particular for functions f that have oscillations that accelerate in a neighborhood of v (e.g., $f(t) = \sin(1/t)$ for $v = 0$).

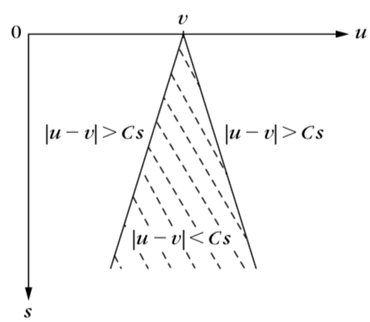


Figure 25: The cone of influence $\mathcal{C}_v$ of an abscissa v consists of the time-scale points (u, s)

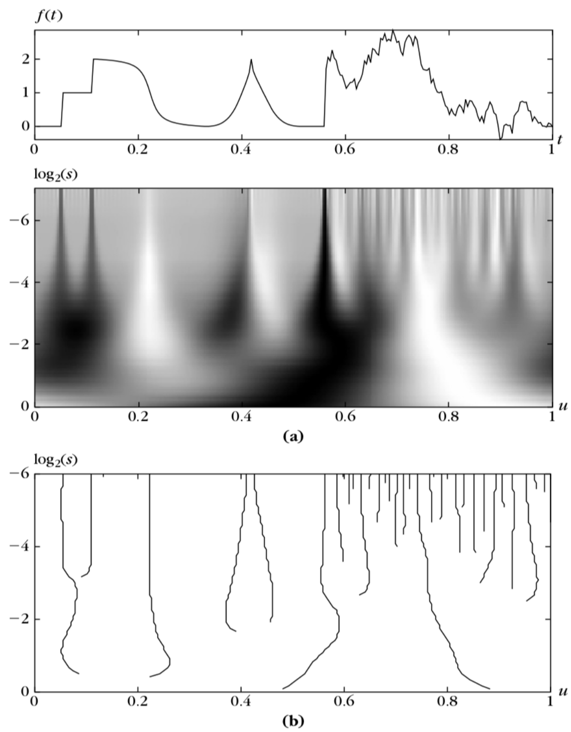
Within the cone $\mathcal{C}_v$ we have $|u-v|/s \leq C$, and so the conditions (38) and (39) of Theorem 5.5 can be written for these points as:

$$|Wf(u, s)| \leq A' s^{\alpha + 1/2}, \quad \forall (u, s) \in \mathcal{C}_v$$

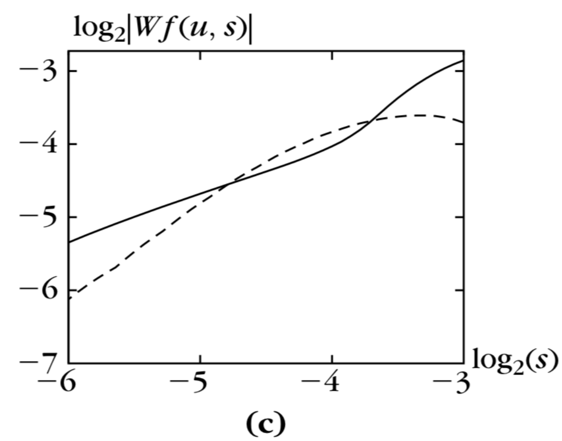
This is equivalent to:

$$\log_2 |Wf(u, s)| \leq \log_2 A' + \left(\alpha + \frac{1}{2}\right) \log_2 s$$

Thus the Lipschitz regularity at v can be estimated by computing the maximum slope of $\log_2 |Wf(u, s)|$ as a function of $\log_2 s$ along the maxima line converging to v. Figure 26 describes an example.



(a) Figure 23 revisited



(b) $\log_2 |Wf(u,s)|$ as a function of $\log_2 s$ along two maxima lines.

Figure 26: Figure (b) plots $\log_2 |Wf(u,s)|$ as a function of $\log_2 s$ along two maxima lines. The solid line corresponds to the maxima line that converges to $v = 0.05$. It has a maximum slope of $\alpha + 1/2 \approx 1/2$, implying that $\alpha = 0$, which is expected since $f(t)$ is discontinuous at $t = 0.05$. The dashed line corresponds to the maxima line converging to $v = 0.42$. Here the maximum slope is $\alpha + 1/2 \approx 1$, indicating that the singularity is Lipschitz 1/2.

In practice variations in a signal $f(t)$ may correspond to smooth singularities, for example due to blur or shadows in an image. In this case, points of rapid transition will technically be $\mathbf{C}^\infty$. However, if the blurring effect is from a Gaussian kernel, we can still get precise measurements on the decay of the wavelet coefficients.

We suppose that in the neighborhood of a sharp transition v, $f(t)$ can be modeled as

$$f(t) = f_0 * g_\sigma(t)$$

where

$$g_\sigma(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/2\sigma^2}$$

If f_0 is uniformly Lipschitz α in a neighborhood of v, then we can relate the decay of the wavelet coefficients to α and σ so long as $\psi = (-1)^n \theta^{(n)}$ for θ a Gaussian.

Theorem 5.11. *Let* $\psi = (-1)^n \theta^{(n)}$ *with*

$$\theta(t) = \lambda e^{-t^2/2\beta^2}$$

If $f = f_0 * g_\sigma$ *and* f_0 *is uniformly Lipschitz* $\alpha \leq n$ *on* $[v-\varepsilon, v+\varepsilon]$*, then there exists* $A > 0$ *such that*

$$|Wf(u,s)| \leq A s^{\alpha+1/2} \left(1 + \frac{\sigma^2}{\beta^2 s^2}\right)^{-(n-\alpha)/2}, \quad \forall\,(u,s) \in [v-\varepsilon, v+\varepsilon] \times (0,\infty)$$

Proof. Using Theorem 5.4 we write the wavelet transform as:

$$Wf(u,s) = s^n \frac{d^n}{du^n}(f * \theta_s)(u) = s^n \frac{d^n}{du^n}(f_0 * g_\sigma * \theta_s)(u)$$

Since g_σ and θ are Gaussians, $g_\sigma * \theta_s$ is also a Gaussian and one calculate its scale as:

$$g_\sigma * \theta_s(t) = \sqrt{\frac{s}{s_0}}\theta_{s_0}(t), \quad s_0 = \sqrt{s^2 + \frac{\sigma^2}{\beta^2}}$$

Therefore we can rewrite the wavelet transform as

$$\begin{aligned} Wf(u,s) &= s^n \sqrt{\frac{s}{s_0}} \frac{d^n}{du^n}(f_0 * \theta_{s_0})(u) \\ &= \left(\frac{s}{s_0}\right)^{n+1/2} s_0^n \frac{d^n}{du^n}(f_0 * \theta_{s_0})(u) \\ &= \left(\frac{s}{s_0}\right)^{n+1/2} Wf_0(u,s_0) \end{aligned}$$

Since f_0 is uniformly Lipschitz α on $[v-\varepsilon, v+\varepsilon]$, Theorem 5.7 proves that there exists $A > 0$ such that

$$|Wf_0(u,s)| \leq A s^{\alpha+1/2}, \quad \forall\,(u,s) \in [v-\varepsilon, v+\varepsilon] \times (0,\infty)$$

Therefore,

$$
\begin{aligned}
|Wf(u,s)| &\le \left(\frac{s}{s_0}\right)^{n+1/2} |Wf_0(u,s_0)| \\
&\le \left(\frac{s}{s_0}\right)^{n+1/2} A s_0^{\alpha+1/2} \\
&= A s^{n+1/2} s_0^{-(n-\alpha)} \\
&= A s^{n+1/2} \left(s^2 + \frac{\sigma^2}{\beta^2}\right)^{-(n-\alpha)/2} \\
&= A s^{\alpha+1/2} \left(1 + \frac{\sigma^2}{\beta^2 s^2}\right)^{-(n-\alpha)/2}
\end{aligned}
$$

□

This theorem relates the wavelet transform decay expected by the Lipschitz α singularity versus what one observes due to the diffusion at the singularity. At large scales $s \gg \sigma/\beta$, the bound is essentially $|Wf(u,s)| \le As^{\alpha+1/2}$ since the second term becomes nearly equal to one. In other words, the larger wavelets do not "feel" the blurring effect. However, for $s \le \sigma/\beta$, the decay is more like $|Wf(u,s)| \le As^{n+1/2}$, which depends upon the number of vanishing moments of the wavelet, not the regularity of the underlying singularity. This is because the blurred signal is in fact $\mathbf{C}^\infty$, and thus the decay at fine scales will necessarily be limited by the finite number of vanishing moments. Figure 27 gives an example.

Exercise 53. Read Section 6.2.1 of *A Wavelet Tour of Signal Processing.*

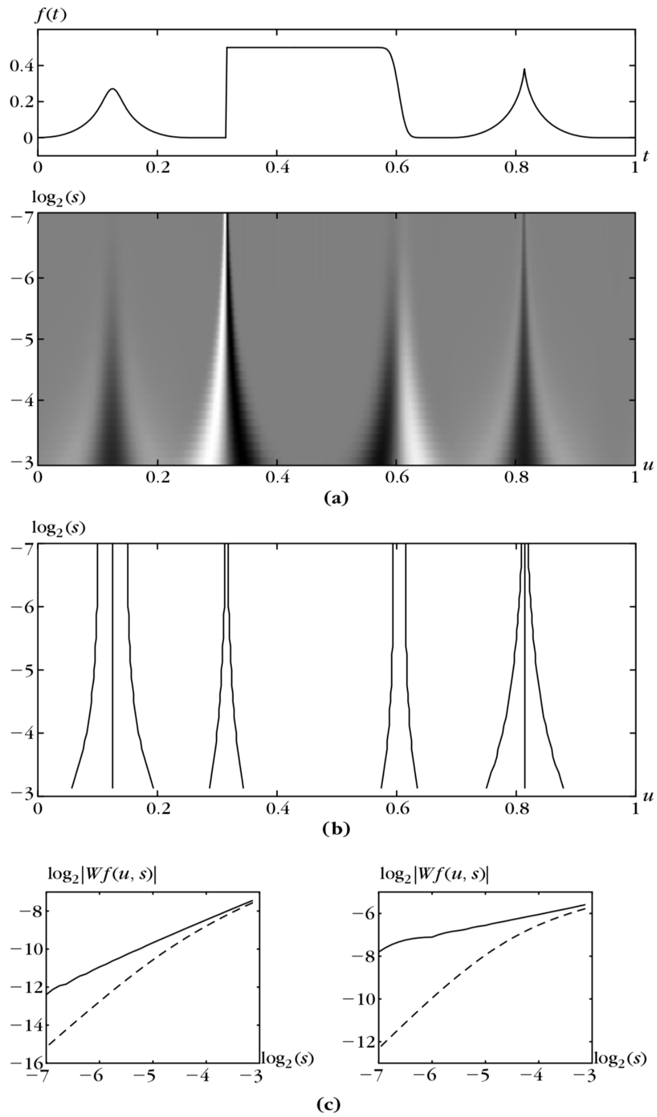


Figure 27: Top: Signal with two types of singularities, a jump discontinuity at $t = 0.35$ and a cusp at $t = 0.81$. Blurred versions of the same singularities are located at $t = 0.60$ and $t = 0.12$, respectively. (a) The wavelet transform $Wf(u,s)$ using a wavelet $\psi = \theta''$, where θ is a Gaussian with variance $\beta = 1$. (b) Modulus maxima lines. (c) Decay of $\log_2 |Wf(u,s)|$ along the maxima lines. The solid and dashed lines on the left correspond to the maxima lines converging to $t = 0.81$ and $t = 0.12$, respectively. The solid and dashed lines on the right correspond to the maxima lines converging to $t = 0.35$ and $t = 0.60$, respectively. Thus the solid lines correspond to the singularities, and the dashed lines correspond to the blurred singularities. Notice that the diffusion modifies the decay for $s \leq \sigma = 2^{-5}$.

References

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