Lecture 13: Wavelet Vanishing Moments

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The next theorem generalizes Theorem 2.15 by relating the decay of the Fourier transform of $f(t)$ to the α regularity of f.

Theorem 5.3. *Suppose that $f \in \mathbf{L}^1(\mathbb{R})$. If*

$$\int_{\mathbb{R}} |\widehat{f}(\omega)|(1 + |\omega|^\alpha)\, d\omega < +\infty \tag{34}$$

then $f \in \mathbf{C}^\alpha(\mathbb{R})$.

Proof. Equation (34) implies that $\widehat{f} \in \mathbf{L}^1(\mathbb{R})$, and so the Fourier inversion formula (2) holds. We use it to prove $f \in \mathbf{L}^\infty(\mathbb{R})$:

$$\begin{aligned} |f(t)| \leq \frac{1}{2\pi}\left|\int_{\mathbb{R}} \widehat{f}(\omega) e^{i\omega t}\, d\omega\right| &\leq \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\omega)|\, d\omega \\ &\leq \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\omega)|(1+|\omega|^\alpha)\, d\omega < \infty \end{aligned}$$

Now suppose that $0 < \alpha < 1$ and show that $f \in \dot{\mathbf{C}}^\alpha(\mathbb{R})$. To do so we need to show there exists $K > 0$ such that

$$|f(t) - f(v)| \leq K|t - v|^\alpha, \quad \forall\, t, v \in \mathbb{R}$$

By the Fourier inversion formula (2) we have that

$$f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\omega) e^{i\omega t}\, d\omega$$

It follows that

$$\frac{|f(t) - f(v)|}{|t - v|^\alpha} \leq \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\omega)| \frac{|e^{i\omega t} - e^{i\omega v}|}{|t - v|^\alpha}\, d\omega$$

For $|\omega| \geq |t - v|^{-1}$,

$$\frac{|e^{i\omega t} - e^{i\omega v}|}{|t - v|^\alpha} \leq \frac{2}{|t - v|^\alpha} \leq 2|\omega|^\alpha \tag{35}$$

On the other hand, for $|\omega| \leq |t - v|^{-1}$, we note that if a function $h \in \mathbf{C}^1(\mathbb{R})$ with bounded derivative then

$$|h(t) - h(v)| \leq K|t - v|, \quad K = \sup_{u \in \mathbb{R}} |h'(u)|$$

Note that $e_\omega \in \mathbf{C}^1(\mathbb{R})$, where $e_\omega(t) = e^{i\omega t}$, and $|e'_\omega(t)| = |\omega|$. Therefore,

$$\frac{|e^{i\omega t} - e^{i\omega v}|}{|t-v|^\alpha} \leq \frac{|\omega||t-v|}{|t-v|^\alpha} = |\omega||t-v|^{1-\alpha} \leq |\omega||\omega|^{\alpha-1} = |\omega|^\alpha \tag{36}$$

Combining (35) and (36), we obtain

$$\frac{|f(t)-f(v)|}{|t-v|^\alpha} \leq \frac{1}{2\pi}\int_{\mathbb{R}} 2|\widehat{f}(\omega)||\omega|^\alpha \, d\omega = K$$

Equation (34) ensures that $K < \infty$, and so $f \in \mathbf{C}^\alpha(\mathbb{R})$.

We now extend the result to $\alpha > 1$, $\alpha \notin \mathbb{Z}$. Let $n = \lfloor \alpha \rfloor$. Theorem 2.15 proves that $f \in \mathbf{C}^n(\mathbb{R})$. Recall that $\widehat{f^{(k)}}(\omega) = (i\omega)^k \widehat{f}(\omega)$. Equation (34) gives:

$$\int_{\mathbb{R}} |\widehat{f^{(k)}}(\omega)|(1+|\omega|^{\alpha-n})\, d\omega = \int_{\mathbb{R}} |\widehat{f}(\omega)|(|\omega|^k + |\omega|^{\alpha-n+k})\, d\omega < \infty$$

Thus by our work above, we have that $f^{(k)} \in \mathbf{C}^{\alpha-n}(\mathbb{R})$ for $k \leq n$, which proves that $f \in \mathbf{C}^\alpha(\mathbb{R})$. □

As we have discussed previously for $\mathbf{C}^n$-smooth functions, the decay of the Fourier transform can only indicate the minimum regularity of $f(t)$. Wavelet transforms characterize both the global and pointwise regularity of functions.

Exercise 45. Read Section 6.1.1 of *A Wavelet Tour of Signal Processing*.

Exercise 46. Consider the function

$$f(t) = t \sin\left(\frac{1}{t}\right)$$

(a) Prove that $f(t)$ is pointwise Lipschitz 1 for all $t \in (-1,1)$.

(b) Prove that $f \in \mathbf{C}^\alpha(-1,1)$ only for $\alpha \leq 1/2$ (*Hint:* Consider the points $t_n = (n + 1/2)^{-1}\pi^{-1}$).

5.1.2 Wavelet Vanishing Moments

Section 6.1.2 of A Wavelet Tour of Signal Processing.

We assume throughout that $\psi(t)$ is a real valued wavelet. A wavelet ψ has n vanishing moments if

$$\int_{\mathbb{R}} t^k \psi(t)\, dt = 0, \quad \forall\, 0 \leq k < n$$

A wavelet ψ with n vanishing moments is orthogonal to polynomials of degree $n-1$.

Suppose now that f is Lipschitz $\alpha < n$ at v, so that

$$f(t) = p_v(t) + \varepsilon_v(t)$$

with $p_v(t)$ a polynomial of degree $n-1$ and

$$|\varepsilon_v(t)| \leq K|t-v|^\alpha$$

We have that

$$Wp_v(u,s) = \int_{\mathbb{R}} p_v(t)\frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)\,dt = \sqrt{s}\int_{\mathbb{R}} p_v(st'+u)\psi(t')\,dt' = 0$$

Therefore,

$$Wf(u,s) = Wp_v(u,s) + W\varepsilon_v(u,s) = W\varepsilon_v(u,s)$$

Thus a wavelet transform with n vanishing moments analyzes $f(t)$ around $t=v$ by ignoring the polynomial approximation of $f(t)$ and focusing on the residual $\varepsilon_v(t)$.

A wavelet ψ has fast decay if

$$\forall\, m \in \mathbb{N},\ \exists\, C_m \text{ such that } |\psi(t)| \leq \frac{C_m}{1+|t|^m}, \quad \forall\, t \in \mathbb{R}$$

The following theorem shows that a wavelet ψ with fast decay and n vanishing moments is the n^{th} derivative of a function $\theta(t)$. The resulting wavelet transform is thus a multiscale differential operator.

Theorem 5.4. *A wavelet $\psi(t)$ with fast decay has n vanishing moments if and only if there exists $\theta(t)$ with a fast decay such that*

$$\psi(t) = (-1)^n\theta^{(n)}(t)$$

Consequently,

$$Wf(u,s) = s^n\frac{d^n}{du^n}(f * \overline{\theta}_s)(u)$$

where $\overline{\theta}_s(t) = s^{-1/2}\theta(-t/s)$. Furthermore, ψ has no more than n vanishing moments if and only if

$$\int_{\mathbb{R}} \theta(t)\,dt \neq 0$$

Proof. Suppose that ψ has fast decay and n vanishing moments. Since ψ has fast decay we must have that $\widehat{\psi} \in \mathbf{C}^\infty(\mathbb{R})$; this follows from Theorem 2.15 by setting $f = \widehat{\psi}$. Thus we can differentiate $\widehat{\psi}(\omega)$ as many times as we like.

Recall that the Fourier transform of $h(t) = (-it)^k\psi(t)$ is $\widehat{h}(\omega) = \widehat{\psi}^{(k)}(\omega)$. It follows that

$$\widehat{\psi}^{(k)}(0) = \int_{\mathbb{R}} (-it)^k\psi(t)\,dt = (-i)^k\int_{\mathbb{R}} t^k\psi(t)\,dt = 0, \quad \forall\, 0 \leq k < n$$

We can therefore write $\widehat{\psi}$ as

$$\widehat{\psi}(\omega) = (-i\omega)^n \widehat{\theta}(\omega)$$

where $\widehat{\theta} \in \mathbf{L}^\infty(\mathbb{R})$ since $\widehat{\psi} \in \mathbf{L}^\infty(\mathbb{R})$. It follows that

$$\psi(t) = (-1)^n \theta^{(n)}(t)$$

The fast decay of $\theta(t)$ is proved with an induction on n. For $n = 1$,

$$\widehat{\psi}(\omega) = -i\omega\widehat{\theta}(\omega) \Longrightarrow \psi(t) = -\theta'(t)$$

It follows that

$$\theta(t) = -\int_{-\infty}^{t} \psi(u)\, du$$

Thus, using the fast decay of $\psi(t)$,

$$|\theta(t)| \leq \int_{-\infty}^{t} |\psi(u)|\, du \leq \int_{-\infty}^{t} \frac{C_m}{1 + |u|^m}\, du \leq \frac{C'_{m-1}}{1 + |t|^{m-1}}, \quad \forall\, m \geq 2$$

Now make the inductive hypothesis that if $\Psi(t)$ is any wavelet with fast decay and

$$\widehat{\Psi}(\omega) = (-i\omega)^k \widehat{\Theta}(\omega), \quad 1 \leq k \leq n$$

then $\Theta(t)$ has fast decay. Consider now a wavelet ψ with fast decay that has $n+1$ vanishing moments, so that $\widehat{\psi}(\omega) = (-i\omega)^{n+1}\widehat{\theta}(\omega)$. Define

$$\widehat{\Theta}(\omega) = -i\omega\widehat{\theta}(\omega) \Longrightarrow \widehat{\psi}(\omega) = (-i\omega)^n \widehat{\Theta}(\omega)$$

By the inductive hypothesis, $\Theta(t)$ has fast decay. But then since $\widehat{\Theta}(\omega) = -i\omega\widehat{\theta}(\omega)$, we can apply the inductive hypothesis again to conclude that $\theta(t)$ has fast decay.

Conversely, suppose that $\psi(t) = (-1)^n \theta^{(n)}(t)$ and $\theta(t)$ has fast decay. Because of the fast decay,

$$|\widehat{\theta}(\omega)| \leq \int_{\mathbb{R}} |\theta(t)|\, dt \leq \int_{\mathbb{R}} \frac{C_m}{1 + |t|^m}\, dt < +\infty, \quad m \geq 2$$

Thus $\widehat{\theta} \in \mathbf{L}^\infty(\mathbb{R})$. The Fourier transform of $\psi(t)$ is

$$\widehat{\psi}(\omega) = (-i\omega)^n \widehat{\theta}(\omega)$$

It follows that $\widehat{\psi}^{(k)}(0) = 0$ for $k < n$. But then

$$\int_{\mathbb{R}} t^k \psi(t)\, dt = i^k \widehat{\psi}^{(k)}(0) = 0, \quad 0 \leq k < n$$

Thus $\psi(t)$ has n vanishing moments.

To test whether $\psi(t)$ has more than n vanishing moments, we compute:

$$\int_{\mathbb{R}} t^n \psi(t)\, dt = i^n \widehat{\psi}^{(n)}(0) = (-i)^n n! \widehat{\theta}(0)$$

Clearly then ψ has no more than n vanishing moments if and only if

$$\widehat{\theta}(0) = \int_{\mathbb{R}} \theta(t)\, dt \neq 0$$

Recall the wavelet transform can be written as

$$Wf(u,s) = f * \overline{\psi}_s(u)$$

where

$$\overline{\psi}_s(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{-t}{s}\right) = \frac{(-1)^n}{\sqrt{s}} \theta^{(n)}\left(-\frac{t}{s}\right) = (-1)^n \overline{\theta}_s^{(n)}(t)$$

A simple calculation also shows that

$$\frac{d^n}{dt^n}\overline{\theta}_s(t) = \frac{1}{s^n}\frac{(-1)^n}{\sqrt{s}}\theta^{(n)}\left(-\frac{t}{s}\right) = \frac{(-1)^n}{s^n}\overline{\theta}_s^{(n)}(t) = \frac{\overline{\psi}_s(t)}{s^n}$$

Therefore $\overline{\psi}_s(t) = s^n (d^n/dt^n)\overline{\theta}_s(t)$. We then have:

$$Wf(u,s) = f * \overline{\psi}_s(u) = s^n f * \theta_s^{(n)}(u) = s^n \frac{d^n}{du^n}(f * \theta)(u)$$

□

If $K = \widehat{\theta}(0) \neq 0$, then the convolution $f * \overline{\theta}_s(t)$ can be interpreted as a weighted average of f with a kernel dilated by s. Theorem 5.4 proves that $Wf(u,s)$ is an n^{th} order derivative of an averaging of f over a domain proportional to s and centered at u. Figure plots $Wf(u,s)$ calculated with $\psi(t) = -\theta'(t)$, where $\theta(t)$ is a Gaussian. Notice how the sign and magnitude of the wavelet coefficients corresponds to the derivative of f averaged over a window of size proportional to s. Compare to Figure 19, which computed $Wf(u,s)$ with the Mexican hat wavelet $\psi(t) = \theta''(t)$ (θ again a Gaussian).

Since $\theta(t)$ has fast decay, once can verify that for any f that is continuous at u,

$$\lim_{s \to 0} f * \frac{1}{\sqrt{s}}\overline{\theta}_s(u) = Kf(u)$$

In the sense of distributions, we write

$$\lim_{s \to 0} \frac{1}{\sqrt{s}}\overline{\theta}_s(t) = K\delta(t)$$

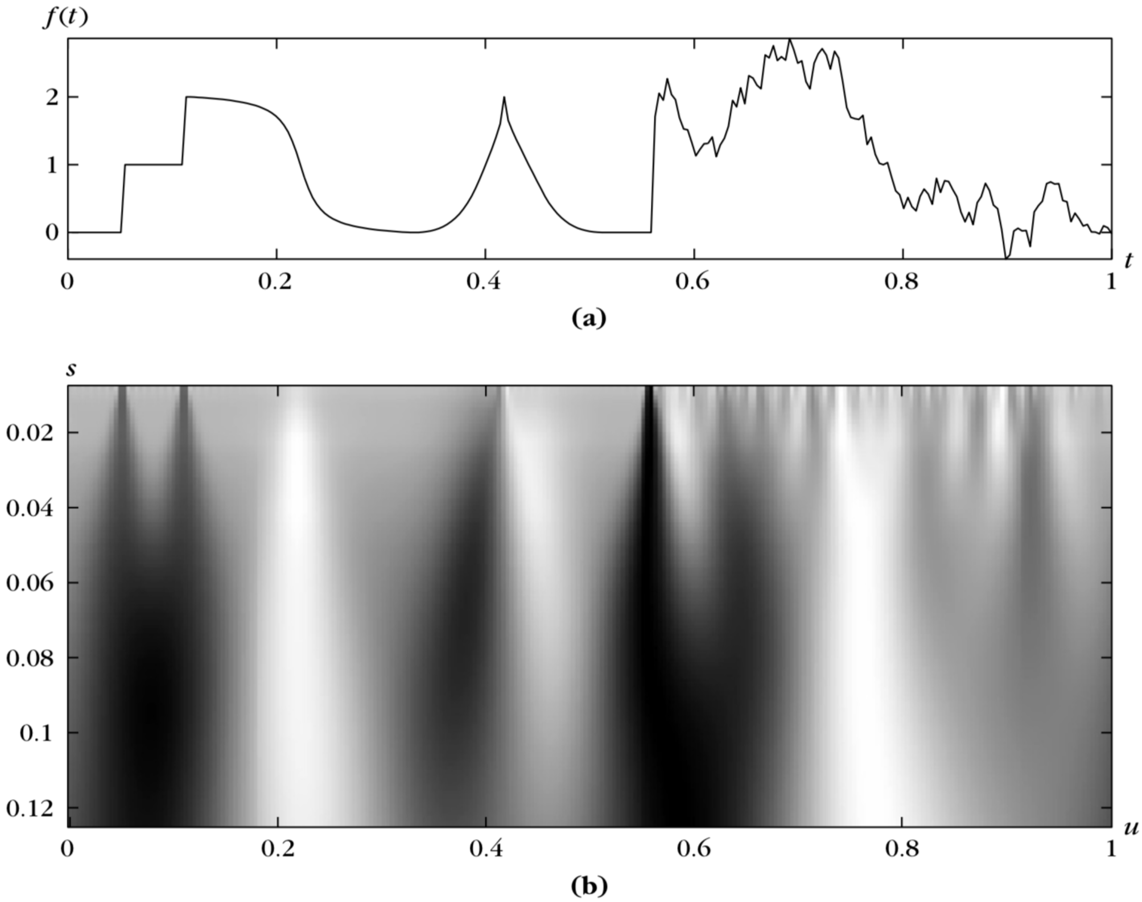

Figure 22: Wavelet transform $Wf(u,s)$ calculated with $\psi = -\theta'$, where θ is a Gaussian, for the signal $f(t)$ shown in (a). Position parameter u and scale s vary, respectively, along the horizontal and vertical axes. (b) Black, gray, and white points correspond to positive, zero, and negative wavelet coefficients. Singularities create large-amplitude coefficients in their cone of influence.

If f is n times continuously differentiable in the neighborhood of u, then using Theorem 5.4,

$$\lim_{s\to 0} \frac{Wf(u,s)}{s^{n+1/2}} = \lim_{s\to 0} \frac{1}{\sqrt{s}} \frac{d^n}{dt^n}(f * \overline{\theta}_s)(u) = \lim_{s\to 0} f^{(n)} * \frac{1}{\sqrt{s}} \overline{\theta}_s(u) = Kf^{(n)}(u) \tag{37}$$

In particular, if $f \in \mathbf{C}^n(\mathbb{R})$, then $|Wf(u,s)| = O(s^{n+1/2})$. This gives us a first relation between the decay of $|Wf(u,s)|$ as $s \to 0$ and the uniform regularity of f. Next we push harder and obtain finer relations.

Exercise 47. Read Section 6.1.2 of *A Wavelet Tour of Signal Processing.*

References

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[5] Karlheinz Gröchenig. *Foundations of Time Frequency Analysis*. Springer Birkhäuser, 2001.