Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 11: Wavelet Ridges

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4.4.2 Wavelet Ridges

Section 4.4.3 of A Wavelet Tour of Signal Processing.

Motivated by the hyperbolic chirp example and the poor performance of the windowed Fourier ridges for this example, we define and study wavelet ridges. We utilize an approximately analytic "wavelet" $\psi(t)$ of the form:

$$\psi(t) = g(t)e^{i\eta t}$$

where the window function $g(t)$ satisfies the same assumptions as in the windowed Fourier case; namely:

- $\operatorname{supp} g = [-1/2, 1/2]$
- $g(t) \geq 0$ so that $|\widehat{g}(\omega)| \leq \widehat{g}(0)$ for all $\omega \in \mathbb{R}$
- $\|g\|_2 = 1$ but also $\widehat{g}(0) = \int g(t)\, dt = \|g\|_1 \approx 1$

Let $\Delta\omega$ be the bandwidth of $\widehat{g}$. If $\eta > \Delta\omega$ then

$$\widehat{\psi}(\omega) = \widehat{g}(\omega - \eta) \ll 1, \quad \forall\, \omega \leq 0$$

Thus $\psi(t)$ is not strictly a wavelet nor is it strictly analytic, but it nearly satisfies both conditions.

Notice that dilated and translated wavelets can be written as:

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) = g_{s,u,\eta/s}(t)e^{-i(\eta/s)u}$$

where

$$g_{s,u,\xi}(t) = \frac{1}{\sqrt{s}}g\left(\frac{t-u}{s}\right)e^{i\xi t}$$

The resulting wavelet transform use time frequency atoms similar to those of the windowed Fourier transform. However, in this case the scale s varies over $(0, +\infty)$ and $\xi = \eta/s$:

$$Wf(u,s) = \langle f, \psi_{u,s}\rangle = \langle f, g_{u,s,\eta/s}\rangle e^{i(\eta/s)u}$$

Theorem 4.5 computes $\langle f, g_{u,s,\xi}\rangle$ when $f(t) = a(t)\cos\theta(t)$. Applying this theorem to the wavelet transform gives:

$$Wf(u,s) = \langle f, g_{u,s,\eta/s}\rangle e^{i(\eta/s)u} = \frac{\sqrt{s}}{2} a(u) e^{i\theta(u)} \left[\widehat{g}(s[\eta/s - \theta'(u)]) + \varepsilon(u,\eta/s)\right]$$

A normalized scalogram computes

$$\widetilde{P}_W f(u,\eta/s) = \frac{|Wf(u,s)|^2}{s} = \frac{1}{4} a(u)^2 \left|\widehat{g}(s[\eta/s - \theta'(u)]) + \varepsilon(u,\eta/s)\right|^2$$

If the error term $\varepsilon(u,\eta/s)$ is negligible, $\widetilde{P}_W f(u,\eta/s)$ obtains its maxima at $(u, \eta/s_u)$ where

$$\frac{\eta}{s_u} = \theta'(u) \Longrightarrow s_u = \frac{\eta}{\theta'(u)}$$

The corresponding points $(u,\eta/s_u)$ are called *wavelet ridges*.

Recall the error term $\varepsilon(u,\eta/s)$ is broken into four components:

$$|\varepsilon(u,\eta/s)| \leq \varepsilon_{a,1}(u,\eta/s) + \varepsilon_{a,2}(u,\eta/s) + \varepsilon_{\theta,2}(u,\eta/s) + \underbrace{\sup_{\omega \geq s\theta'(u)} |\widehat{g}(\omega)|}_{\text{(iv)}}$$

At the ridge points $(u,\eta/s_u)$ the first error term $\varepsilon_{a,1}$ and the fourth error term can be made negligible if the bandwidth $\Delta\omega$ satisfies

$$\Delta\omega \leq s_u \theta'(u) \Longrightarrow \Delta\omega \leq \eta$$

but this was assumed from the start in order to make ψ an approximately analytic wavelet, so these two error terms are guaranteed to be small by the choice of the wavelet. Using Theorem 4.5, the second order terms at the ridge points are bounded as:

$$\varepsilon_{a,2}(u,\eta/s_u) \leq \sup_{|t-u|\leq s_u/2} \frac{s_u^2 |a''(t)|}{|a(u)|} = \sup_{|t-u|\leq \eta/2\theta'(u)} \frac{\eta^2}{\theta'(u)^2} \frac{|a''(t)|}{|a(u)|}$$

and

$$\varepsilon_{\theta,2}(u,\eta/s_u) \leq \sup_{|t-u|\leq s_u/2} s_u^2 |\theta''(t)| = \sup_{|t-u|\leq \eta/2\theta'(u)} \frac{\eta^2}{\theta'(u)^2} |\theta''(t)|$$

Thus since $\theta'(u)$ is in the denominator, we see that if the instantaneous frequency is small, $a'(u)$ and $\theta'(u)$ must have slow variations (i.e., $a''(u)$ and $\theta''(u)$ need to be small), but $a'(u)$ and $\theta'(u)$ are allowed to vary much more quickly when the instantaneous frequency is large.

Now turn to our more general signal model:

$$f(t) = \sum_{k=1}^{K} a_k(t) \cos\theta_k(t)$$

Recall that to separate the K instantaneous frequencies we require that

$$\widehat{g}(s_u^k[\theta_k'(u) - \theta_l'(u)]) \ll 1, \quad \forall\, k \neq l, \quad \frac{\eta}{s_u^k} = \theta_k'(u)$$

which can be obtained if

$$\Delta\omega \leq s_u^k|\theta_k'(u) - \theta_l'(u)| = \frac{\eta|\theta_k'(u) - \theta_l'(u)|}{\theta_k'(u)}, \quad k \neq l$$

Under the assumptions on the window g, the primary free parameter one has is the frequency η. There is a tension between on the one hand wanting to make η large relative to the bandwidth, so that the wavelet is nearly analytic, the error terms $\varepsilon_{a,1}$ and (iv) are small, and so multiple instantaneous frequencies are separated; however, the second order error terms $\varepsilon_{a,2}$ and $\varepsilon_{\theta,2}$ may blow up if η is made too large.

Let us now return to the examples of the linear and hyperbolic chirps. We start with the sum of two hyperbolic chirps, which the windowed Fourier transform had trouble analyzing:

$$f(t) = a_1 \cos\left(\frac{\alpha_1}{\beta_1 - t}\right) + a_2 \cos\left(\frac{\alpha_2}{\beta_2 - t}\right)$$

In this case $\theta_k(t) = \alpha_k/(\beta_k - t)$ and $\theta_k'(t) = \alpha_k/(\beta_k - t)^2$. Since the amplitudes a_1 and a_2 are constant, the second order term $\varepsilon_{a,2}(u, s_u) = 0$. The other second order error term is bounded as:

$$\begin{aligned}
\varepsilon_{\theta,2}(u, s_u) &\leq \max_{k=1,2} \sup_{|t-u|\leq\eta/2\theta_k'(u)} \eta^2 \frac{|\theta_k''(t)|}{\theta'(u)^2} \\
&\leq \max_{k=1,2} \sup_{|t-u|\leq\eta(\beta_k-u)^2/2\alpha_k} \eta^2 \frac{2\alpha_k}{(\beta_k - t)^3} \frac{(\beta_k - t)^4}{\alpha_k^2} \\
&\leq \max_{k=1,2} \sup_{|t-u|\leq\eta(\beta_k-u)^2/2\alpha_k} \eta^2 \frac{2(\beta_k - t)}{\alpha_k}
\end{aligned}$$

This error term will be small if

$$\eta^2 \ll \frac{\alpha_k}{\beta_k - t}$$

This will be the case if, for example, $t \in [0, \beta_k)$ and $\eta \ll \sqrt{\alpha_k/\beta_k}$. Figure 16 illustrates how the wavelet ridges successfully follow the instantaneous frequencies of the two hyperbolic chirps.

Now let us go back to the two linear chirps signal

$$f(t) = a_1 \cos(bt^2 + ct) + a_2 \cos(bt^2)$$

which has frequencies $\theta_1(t) = bt^2 + ct$ and $\theta_2(t) = bt^2$. We thus have

$$\frac{|\theta_1'(u) - \theta_2'(u)|}{\theta_1'(u)} = \frac{|c|}{2bt} \to 0 \text{ as } t \to +\infty$$

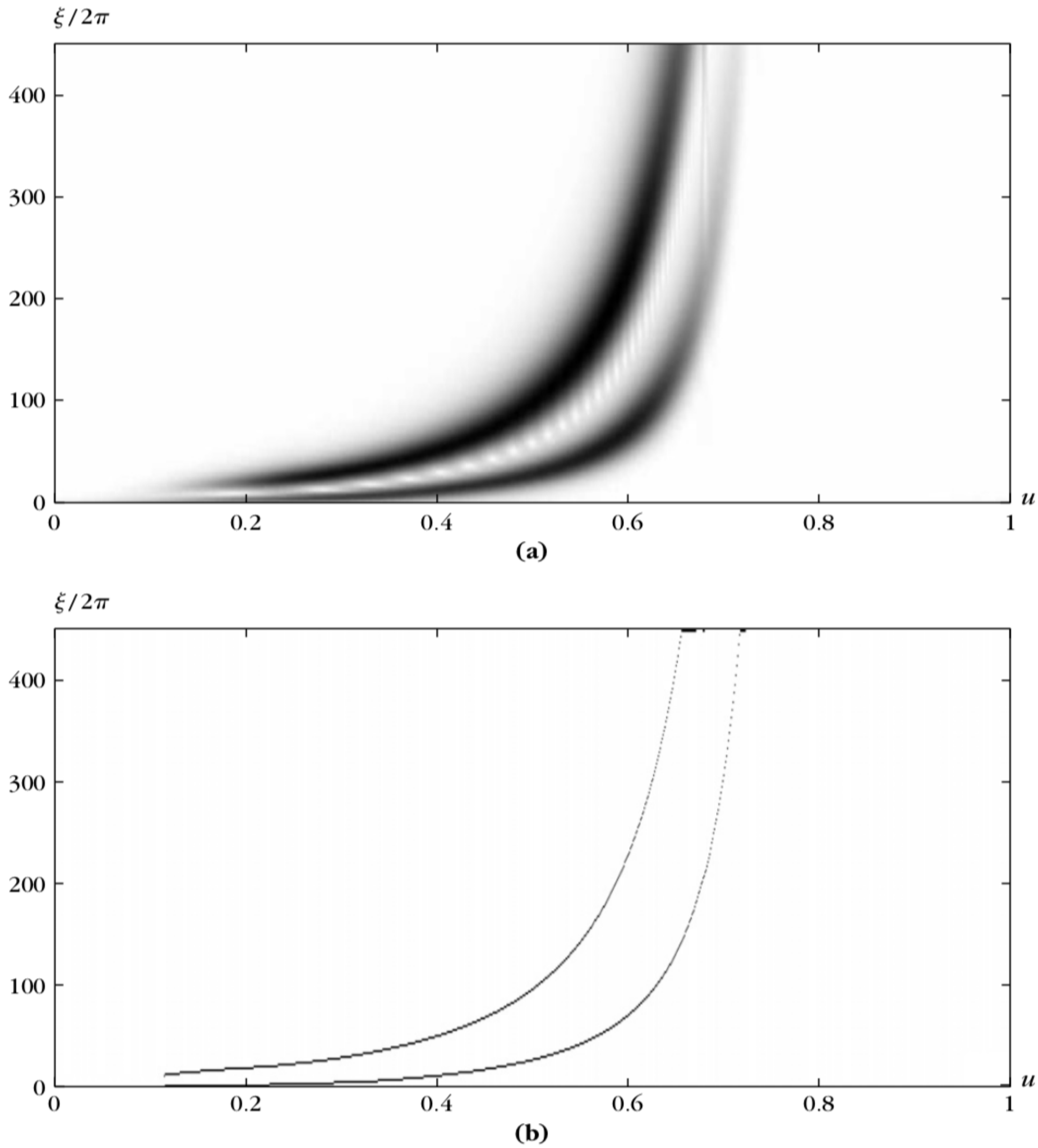

Figure 16: Analysis of a signal consisting of two hyperbolic chirps. (a) Normalized scalogram $\widetilde{P}_W f(u, \eta/s)$; (b) Wavelet ridges.

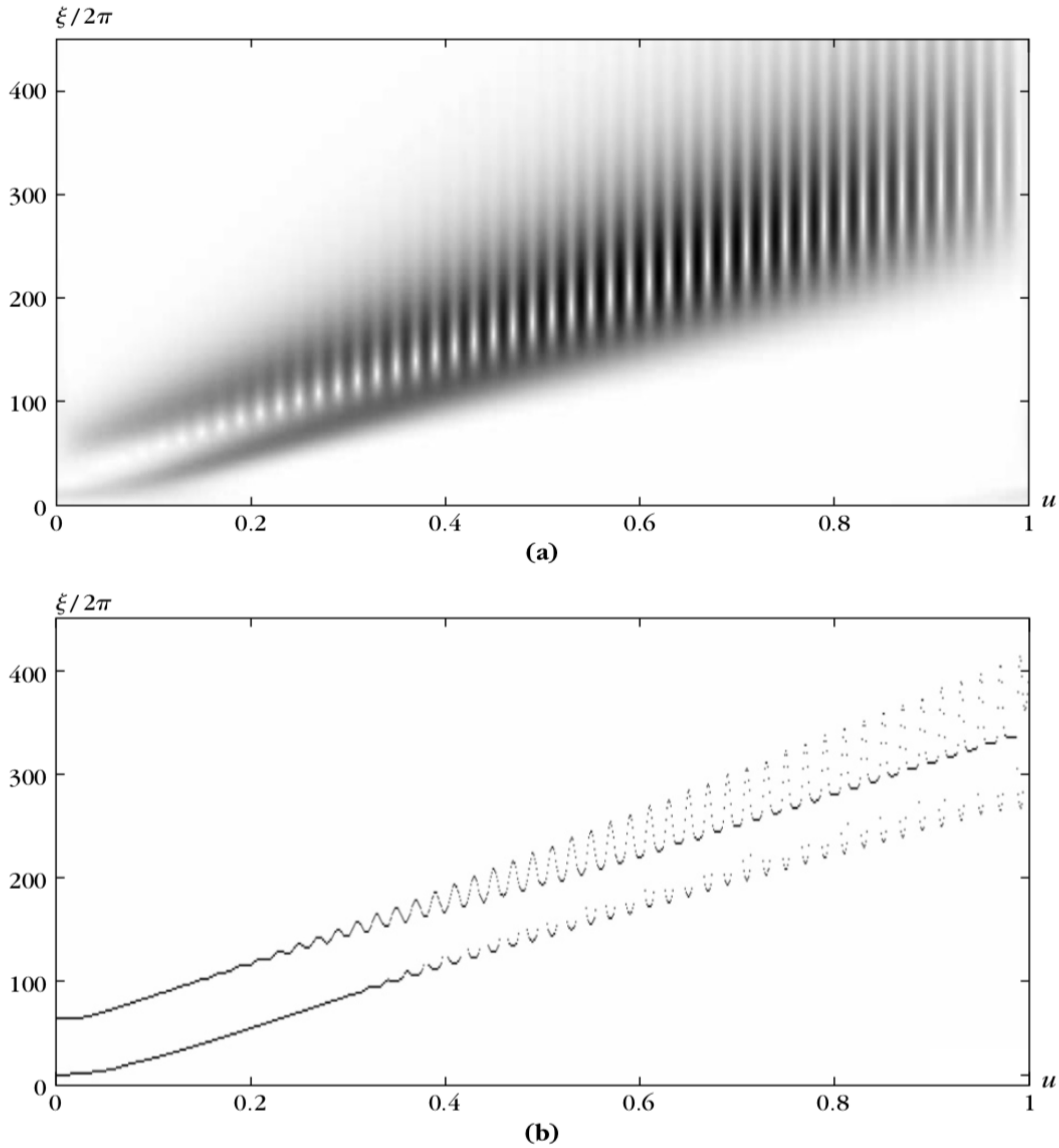

Figure 17: Analysis of a signal consisting of two linear chirps. (a) Normalized scalogram $\widetilde{P}_W f(u, \eta/s)$; (b) Wavelet ridges.

Thus for some finite t we will not be able to separate the instantaneous frequencies because of interferences. Figure 17 illustrates this phenomenon, as for large t the ridges follow the interference patterns rather than the instantaneous frequencies.

The take home message is that better is more sparse. This is true of course for compression, where sparser representations require less memory to store. But the linear and hyperbolic chirp examples show that sparsity also means we have found a time frequency transform that has a resolution adapted to the time frequency properties of the signal, in which case the number of ridge points is small. Conversely, if signal structures do not match our dictionary of time frequency atoms, then their energy will diffuse over many such atoms which produces more ridge points.

Exercise 39. Read Section 4.4.3 of *A Wavelet Tour of Signal Processing.*

Exercise 40. Adapt your windowed Fourier ridge code from Exercise 35 to compute the normalized scalogram $\widetilde{P}_W f(u, \eta/s)$ and corresponding wavelet ridges. Test your code on the sum of two linear chirps and the sum of two hyperbolic chirps. Turn in plots of your wavelet ridges. Do you get something similar to the plots in Figures 16 and 17?

References

[1] Stéphane Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*. Academic Press, 3rd edition, 2008.

[2] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*. Princeton Lectures in Analysis. Princeton University Press, 2003.

[3] John J. Benedetto and Matthew Dellatorre. Uncertainty principles and weighted norm inequalities. *Contemporary Mathematics*, 693:55–78, 2017.

[4] Yves Meyer. *Wavelets and Operators*, volume 1. Cambridge University Press, 1993.

[5] Karlheinz Gröchenig. *Foundations of Time Frequency Analysis*. Springer Birkhäuser, 2001.