Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 09: Windowed Fourier Ridges

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4.3.2 Windowed Fourier Ridges

Section 4.4.2 of A Wavelet Tour of Signal Processing.

We are going to use the windowed Fourier transform, and in particular the local maxima of the windowed Fourier transform, to isolate individual amplitudes $a_k(t)$ and instantaneous frequencies $\theta_k'(t)$ as in the signal model (21), repeated here:

$$f(t) = \sum_{k=1}^{K} a_k(t) \cos \theta_k(t)$$

We make some additional assumptions on the real symmetric window $g(t)$. We suppose that:

- $\operatorname{supp} g = [-1/2, 1/2]$
- $g(t) \geq 0$ so that $|\widehat{g}(\omega)| \leq \widehat{g}(0)$ for all $\omega \in \mathbb{R}$
- $\|g\|_2 = 1$ but also $\widehat{g}(0) = \int g(t)\, dt = \|g\|_1 \approx 1$

For a scale σ set

$$g_\sigma(t) = \sigma^{-1/2} g(\sigma^{-1} t)$$

Note that

$$\operatorname{supp} g_\sigma = [-\sigma/2, \sigma/2] \quad \text{and} \quad \|g_\sigma\|_2 = 1$$

We define the windowed Fourier transform with the scale parameter σ as:

$$S_\sigma f(u, \xi) = \int_{\mathbb{R}} f(t) g_\sigma(t - u) e^{-i\xi t}\, dt$$

The next theorem relates $S_\sigma f(u, \xi)$ to the instantaneous frequency of $f(t)$.

Theorem 4.5. *Let $f(t) = a(t) \cos \theta(t)$. If $\xi \geq 0$, then*

$$S_\sigma f(u, \xi) = \frac{\sqrt{\sigma}}{2} a(u) e^{i[\theta(u) - \xi u]} \Big(\widehat{g}(\sigma[\xi - \theta'(u)]) + \varepsilon(u, \xi) \Big)$$

where

$$|\varepsilon(u,\xi)| \leq \varepsilon_{a,1}(u,\xi) + \varepsilon_{a,2}(u,\xi) + \varepsilon_{\theta,2}(u,\xi) + \sup_{|\omega|\geq\sigma\theta'(u)} |\widehat{g}(\omega)|$$

with

$$\varepsilon_{a,1}(u,\xi) \leq \frac{\sigma|a'(u)|}{|a(u)|}$$

and

$$\varepsilon_{a,2}(u,\xi) \leq \sup_{|t-u|\leq\sigma/2} \frac{\sigma^2|a''(t)|}{|a(u)|}$$

Furthermore, if $\sigma|a'(u)||a(u)|^{-1} \leq 1$*, then*

$$\varepsilon_{\theta,2}(u,\xi) \leq \sup_{|t-u|\leq\sigma/2} \sigma^2|\theta''(t)|$$

And finally, if $\xi = \theta'(u)$*, then*

$$\varepsilon_{a,1}(u,\xi) = \frac{\sigma|a'(u)|}{|a(u)|}|\widehat{g}'(2\sigma\theta'(u))|$$

We omit the proof, which is given in pages 119–122 of *A Wavelet Tour of Signal Processing*. If we can neglect the error term $\varepsilon(u,\xi)$, then we will see that $S_\sigma f(u,\xi)$ enables us to measure $a(u)$ and $\theta'(u)$. This will be the case if $a(t)$ and $\theta(t)$ vary slowly. In particular, $\varepsilon_{a,1}$ is small if $a(t)$ varies slowly over the whole real line, while $\varepsilon_{a,2}$ and $\varepsilon_{\theta,2}$ only require the second derivatives of $a(t)$ and $\theta(t)$ to be small over an interval of length equal to the support of the window g. The fourth part of the error term is small if

$$\Delta\omega \leq \sigma\theta'(u) \tag{22}$$

where recall $\Delta\omega$ is the bandwidth of g.

Let us now suppose that the error term can disregarded, so that

$$S_\sigma f(u,\xi) \approx \frac{\sqrt{\sigma}}{2}a(u)e^{i[\theta(u)-\xi u]}\widehat{g}(\sigma[\xi - \theta'(u)])$$

Since the maximum of $|\widehat{g}(\omega)|$ is at $\omega = 0$, we see that for each u the spectrogram $P_S f(u,\xi) = |S_\sigma f(u,\xi)|^2$ is maximum at $\xi_u = \theta'(u)$. These time frequency points (u,ξ_u), which form curves in the time frequency plane, are called *ridges*. At ridge points we have:

$$S_\sigma f(u,\xi_u) = S_\sigma f(u,\theta'(u)) = \frac{\sqrt{\sigma}}{2}a(u)e^{i[\theta(u)-u\theta'(u)]}(\widehat{g}(0) + \varepsilon(u,\theta'(u)))$$

If the bandwidth satisfies (22), then Theorem 4.5 shows that the $\varepsilon_{a,1}(u,\xi)$ error term is negligible, since in this case $|\widehat{g}'(2\sigma\theta'(u))|$ will be negligible.

We can calculate the amplitude from the ridges as well:

$$a(u) \approx \frac{2|S_\sigma f(u,\theta'(u))|}{\sqrt{\sigma}|\widehat{g}(0)|}$$

if the error term $\varepsilon(u, \theta'(u))$ is small.

The spectrogram computes the instantaneous frequency by computing the magnitude of $S_\sigma(u, \xi)$ along the ridges. Another way to calculate the instantaneous frequency is to look at the phase of $S_\sigma f(u, \xi)$ along the ridges. Let $\Theta_S f(u, \xi)$ be the complex phase of $S_\sigma f(u, \xi)$, which again if the error term can be disregarded, is just:

$$\Theta_S f(u, \xi) \approx \theta(u) - \xi u$$

It follows that

$$\frac{\partial \Theta_S f(u, \xi)}{\partial u} = \theta'(u) - \xi$$

and thus the instantaneous frequency can be computed by estimating this partial derivative and solving for its zeros.

Consider now a signal model

$$f(t) = \sum_{k=1}^{K} a_k(t) \cos \theta_k(t)$$

where $a_k(t)$ and $\theta'_k(t)$ have small variations over intervals of size σ and $\sigma\theta'_k(t) \geq \Delta\omega$ (in other words, we can neglect the error term). Since the windowed Fourier transform is linear, we have:

$$S_\sigma f(u, \xi) \approx \frac{\sqrt{\sigma}}{2} \sum_{k=1}^{K} a_k(u) e^{i[\theta_k(u) - \xi u]} \widehat{g}(\sigma[\xi - \theta'_k(u)])$$

We can distinguish between the K different instantaneous frequencies if

$$\widehat{g}(\sigma[\theta'_k(u) - \theta'_l(u)]) \ll 1, \quad \forall\, u \in \mathbb{R}, \;\; k \neq l \tag{23}$$

We can obtain this condition if the bandwidth of g satisfies

$$\Delta\omega \leq \sigma|\theta'_k(u) - \theta'_l(u)|, \quad \forall\, u \in \mathbb{R}, \;\; k \neq l$$

In this case, when $\xi = \theta'_l(u)$, we have

$$\begin{aligned} S_\sigma f(u, \theta'_l(u)) &\approx \frac{\sqrt{\sigma}}{2} \left(a_l(u) e^{i[\theta_l(u) - u\theta'_l(u)]} \widehat{g}(0) + \underbrace{\sum_{k \neq l} a_k(u) e^{i[\theta_k(u) - u\theta'_l(u)]} \widehat{g}(\sigma[\theta'_l(u) - \theta'_k(u)])}_{\ll 1} \right) \\ &\approx \frac{\sqrt{\sigma}}{2} a_l(u) e^{i[\theta_l(u) - u\theta'_l(u)]} \widehat{g}(0) \end{aligned}$$

and thus we can estimate the instantaneous frequency $\theta'_l(u)$ and corresponding amplitude $a_l(u)$. Notice that the ridge points are distributed along the K time frequency curves $\{(u, \theta'_k(u)) \,:\, u \in \mathbb{R}, \;\; 1 \leq k \leq K\}$. So long as these curves remain well separated (as measured by (23)), we will be able to recover the instantaneous frequencies. However, if the curves get too close, or even worse intersect, then the windowed Fourier transform will have interference and the ridge pattern will be destroyed in that neighborhood.

We have already seen that along the ridge points the error term $\varepsilon_{a,1}(u,\xi)$ is negligible if the bandwidth $\Delta\omega$ is small enough. But we still need make sure the error terms $\varepsilon_{a,2}(u,\xi)$ and $\varepsilon_{\theta,2}(u,\xi)$ are small, which means from Theorem 4.5 we need:

$$\varepsilon_{a,2}(u,\xi) \leq \max_k \sup_{|t-u|\leq\sigma/2} \frac{\sigma^2|a_k''(t)|}{|a_k(u)|} \ll 1$$

and

$$\varepsilon_{\theta,2}(u,\xi) \leq \max_k \sup_{|t-u|\leq\sigma/2} \sigma^2|\theta_k''(t)| \ll 1$$

These place a condition on σ in which we would like to make σ small. However, recall that to make $\varepsilon_{a,1}(u,\xi)$ small at the ridge points and the fourth part of the error term small, we needed

$$\Delta\omega \leq \sigma\theta_k'(u)$$

which means we would like to make σ large. Since $\operatorname{supp} g_\sigma = [-\sigma/2, \sigma/2]$, this means we need to carefully select the window size. Notice how this leads to a tradeoff between localization in time and localization in frequency.

Let us now consider some examples. A linear chirp is of the form:

$$\widetilde{f}(t) = a\cos(bt^2 + ct)$$

It is called linear because its instantaneous frequency is $\theta'(t) = 2bt + c$. Suppose we have a signal consisting of two linear chirps:

$$f(t) = a_1\cos(bt^2 + ct) + a_2\cos(bt^2)$$

To distinguish these two linear chirps, we need our window g to have bandwidth $\Delta\omega$ satisfying

$$\Delta\omega \leq \sigma|\theta_1'(t) - \theta_2'(t)| = \sigma|c|$$

Since the amplitudes are constant, the error term $\varepsilon_{a,2}$ is zero. However, $\varepsilon_{\theta,2}(u,\xi)$ places an upper bound on the time support, which is:

$$\sigma^2|\theta_k''(u)| = 2b\sigma^2 \ll 1, \quad k = 1,2$$

Combining the previous two inequalities we get:

$$\frac{\Delta\omega}{c} \leq \sigma \ll \frac{1}{\sqrt{b}} \Longrightarrow \Delta\omega \ll \frac{c}{\sqrt{b}}$$

Figure 11 illustrates an example.

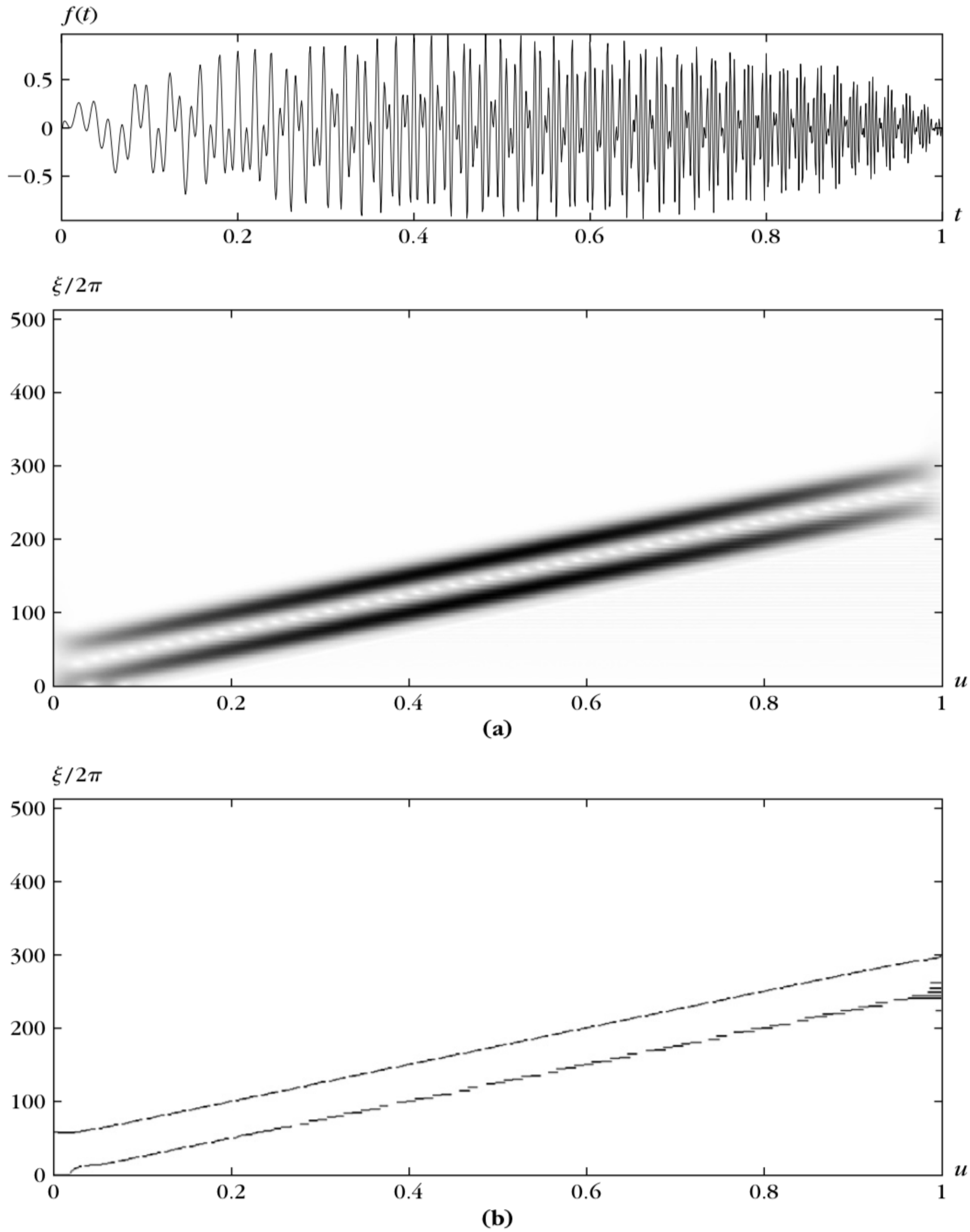

Figure 11: Top: Sum of two parallel linear chirps. Middle: Spectrogram. Bottom: Windowed Fourier ridges.

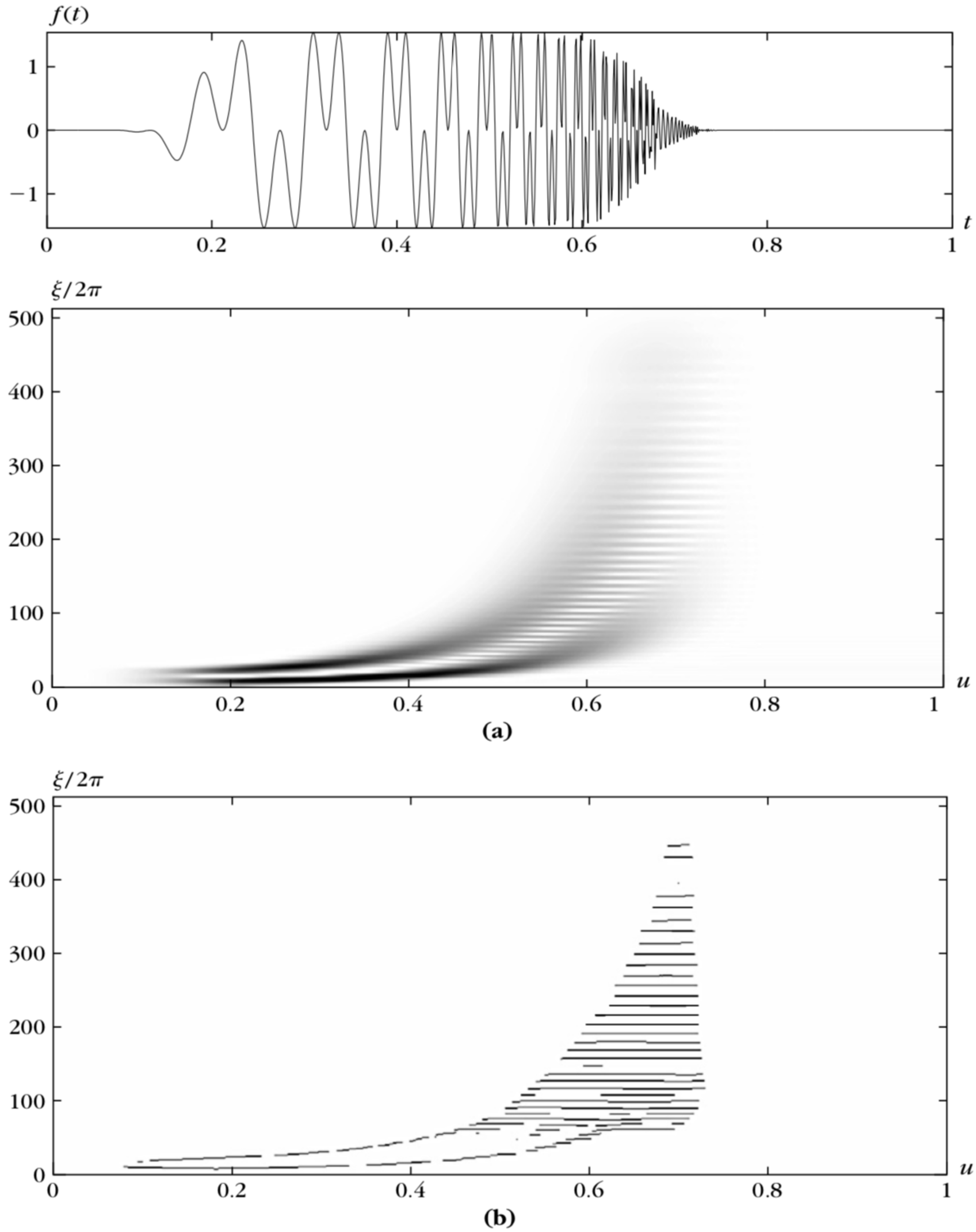

Figure 12: Top: Sum of two hyperbolic chirps. Middle: Spectrogram. Bottom: Windowed Fourier ridges.

Now consider a hyperbolic chirp, which has the form:

$$\widetilde{f}(t) = a \cos\left(\frac{\alpha}{\beta - t}\right)$$

When $t < \beta$, it has instantaneous frequency

$$\theta'(t) = \frac{\alpha}{(\beta - t)^2}$$

which varies quickly as t approaches β. Indeed, as $t \to \beta$ we have $\theta'(t) \to +\infty$, which means that the instantaneous frequency increases to $+\infty$ in a finite amount of time. This is a problem for the windowed Fourier transform because it has a fixed scale in time, and so cannot resolve the fast high frequency changes of the hyperbolic chirp. More precisely, for the error term $\varepsilon_{\theta,2}(u,\xi)$ in Theorem 4.5, we have $\varepsilon_{\theta,2}(u,\xi) \leq \sigma^2|\theta''(u)|$ but for the hyperbolic chirp,

$$\sigma^2|\theta''(u)| = \frac{\sigma^2\alpha}{|\beta - u|^3} > 1, \quad \forall\, |u - \beta| < (\sigma^2\alpha)^{1/3}$$

Therefore the error term is uncontrolled, which leads to a lot of interference in the time frequency response $S_\sigma f(u,\xi)$. Figure 12 illustrates the point for the sum of two hyperbolic chirps,

$$f(t) = a_1 \cos\left(\frac{\alpha_1}{\beta_1 - t}\right) + a_2 \cos\left(\frac{\alpha_2}{\beta_2 - t}\right)$$

Exercise 33. Read Section 4.4.2 of *A Wavelet Tour of Signal Processing*.

Exercise 34. Now we are going to use your windowed Fourier transform code to reproduce some results from the book.

(a) Read Example 4.5 (p. 94) of *A Wavelet Tour of Signal Processing* and determine what the signal is (write it out analytically). Then compute the windowed Fourier transform and corresponding spectrogram, and recreate something similar to Figure 4.3(a). Provide a plot of your spectrogram.

(b) Consider the signal

$$f(t) = a_1 \cos(bt^2 + ct) + a_2 \cos(bt^2)$$

which consists of two real valued linear chirps. Compute the windowed Fourier transform and spectrogram of $f(t)$. Can you find a window g and parameters a_1, a_2, b, c such that you can recreate something similar to Figure 4.13(a)? Provide a plot of your spectrogram. Note: Unlike Exercise 32(b) in which you sampled the single linear chirp on $[-N/2, N/2)$, here sample it on $[0, N)$ so the instantaneous frequency is monotonic.

(c) Consider the signal

$$f(t) = a_1 \cos\left(\frac{\alpha_1}{\beta_1 - t}\right) + a_2 \cos\left(\frac{\alpha_2}{\beta_2 - t}\right)$$

which consists of two hyperbolic chirps. Select parameters $a_1, a_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ and compute the windowed Fourier transform and spectrogram of $f(t)$. Do you get something like Figure 4.14(a)? Provide a plot of your spectrogram.

Exercise 35. We are going to compute numerically windowed Fourier ridges.

(a) Take your windowed Fourier code and add in code to estimate the Fourier ridges by estimating the local maxima of $P_S f(u,\xi) = |S_\sigma f(u,\xi)|^2$.

(b) Now write code to estimate Fourier ridges using the alternate approach, which was to let $\Theta_S f(u,\xi)$ be the complex phase of $S_\sigma f(u,\xi)$, and to solve for ξ such that

$$\frac{\partial \Theta_S f}{\partial u}(u,\xi) = 0$$

(c) Test your code by computing the windowed Fourier ridges of the signals from Exercise 34. Do you get results similar to those from Figures 4.12, 4.13(b), 4.14(b) in *A Wavelet Tour of Signal Processing*? Turn in your plots of the ridges.

Exercise 36. (a) Let $f(t) = \cos(a\cos(bt))$. We want to compute precisely the instantaneous frequency of $f(t)$ from the ridges of its windowed Fourier transform. Find a necessary condition on the window support as a function of a and b.

(b) Now let $f(t) = \cos(a\cos(bt)) + \cos(a\cos(bt) + ct)$. Find a condition on a, b and c in order to measure both instantaneous frequencies with the ridges of a windowed Fourier transform.

(c) Verify your calculations for (a) and (b) numerically using your windowed Fourier ridge code from the previous exercise. Turn in plots of the spectrogram for (a) and (b) and plots of the ridges for (a) and (b).

References

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[5] Karlheinz Gröchenig. *Foundations of Time Frequency Analysis*. Springer Birkhäuser, 2001.