6.2 Translation Invariant Dyadic Wavelet Transform

Section 5.2 of A Wavelet Tour of Signal Processing.

Recall that a continuous wavelet transform computes

$$Wf(u,s) = \langle f, \psi_{u,s}\rangle = f * \bar{\psi}_s(u), \quad \forall (u,s) \in \mathbb{R} \times (0,\infty) \tag{63}$$

where

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \quad \text{and} \quad \bar{\psi}_s(t) = \frac{1}{\sqrt{s}}\psi^*\left(-\frac{t}{s}\right)$$

The operator W, as defined in (63), does not define an analysis operator of a semi-discrete frame because the scale parameter s takes values over the entire interval $(0,\infty)$, which is not discrete.

A semi-discrete wavelet frame is generated by sampling the scale parameter s along an exponential sequence $\{a^j\}_{j\in\mathbb{Z}}$ for some $a > 1$. In many applications (but not all!), we take $a = 2$. In this case the generating family is $\{\psi_j\}_{j\in\mathbb{Z}}$ with

$$\psi_j(t) = 2^{-j}\psi(2^{-j}t)$$

and the translation invariant dictionary is given by:

$$\mathcal{D} = \{\psi_{u,j}\}_{u\in\mathbb{R},\, j\in\mathbb{Z}}, \quad \psi_{u,j}(t) = \psi_j(t-u) = 2^{-j}\psi(2^{-j}(t-u))$$

The resulting analysis operator defines the dyadic wavelet transform:

$$Wf(u,j) = \langle f, \psi_{u,j}\rangle = f * \bar{\psi}_j(u), \quad \bar{\psi}_j(t) = 2^{-j}\psi^*(-2^{-j}t)$$

Notice that rather than normalizing the dilated wavelets by $2^{-j/2}$, which would be analogous to the normalization $s^{-1/2}$ in the continuous wavelet transform, we normalize by 2^{-j}. This is to simplify the following presentation. It simply means that the normalization preserves the $\mathbf{L}^1$ norm of ψ as opposed to the $\mathbf{L}^2$ norm, that is, $\|\psi_j\|_1 = \|\psi\|_1$. Notice as well that $\widehat{\psi}_j(\omega) = \widehat{\psi}(2^j\omega)$ with this normalization.

Applying Theorem 44 shows that $\mathcal{D}$ is a semi-discrete frame if and only if there exists $0 < A \leq B < 0$ such that

$$A \leq \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 \leq B, \quad \forall \omega \in \mathbb{R} \setminus \{0\} \tag{64}$$

In this case $W : \mathbf{L}^2(\mathbb{R}) \to \boldsymbol{\ell}^2(\mathbf{L}^2(\mathbb{R}))$ when the scales are restricted to $s = 2^j$. Notice that if ψ is a complex analytic wavelet (meaning that $\widehat{\psi}(\omega) = 0$ for all $\omega \leq 0$), then it is impossible for (64) to hold. We will come back to this in a bit. For now assume that ψ is a real valued wavelet. The standard semi-discrete frame condition, which is equivalent to (64), is written as:

$$A\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}} \|f * \bar{\psi}_j\|_2^2 \leq B\|f\|_2^2$$

Equation (64) shows that if the frequency axis is completely covered by dilated dyadic wavelets, then a dyadic wavelet transform defines a complete and stable representation of $f \in \mathbf{L}^2(\mathbb{R})$; see Figure 28.

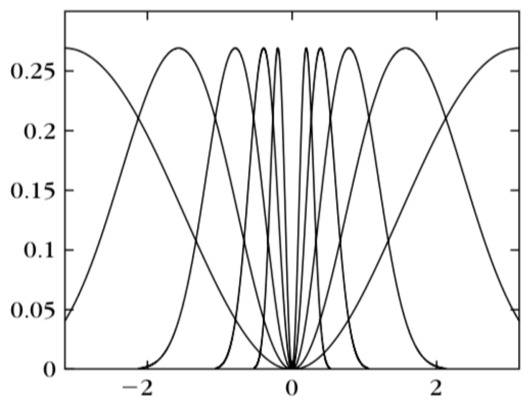

Figure 28: The squared Fourier transform modulus $|\widehat{\psi}(2^j\omega)|^2$ of a real valued spline wavelet, for $1 \leq j \leq 5$ and $\omega \in [-\pi, \pi]$.

In the case of complex analytic wavelets, one option is to use a larger set of generating wavelets given by:

$$\{\psi_{j,\epsilon}\}_{j\in\mathbb{Z},\,\epsilon\in\{1,-1\}}, \quad \psi_{j,\epsilon}(t) = 2^{-j}\psi(\epsilon 2^{-j}t)$$

In this case for suitably chosen wavelets it is possible for (64) to hold. However, it is unnecessary to double the number of generating wavelets as in the above. Indeed, we instead replace (64) with

$$2A \leq \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 + \sum_{j\in\mathbb{Z}} |\widehat{\psi}(-2^j\omega)|^2 \leq 2B, \quad \forall\, \omega \in \mathbb{R}\setminus\{0\} \tag{65}$$

which, due to the wavelet ψ being complex analytic, is equivalent to

$$2A \leq \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 \leq 2B, \quad \forall\, \omega \in (0,\infty)$$

Let $f \in \mathbf{L}^2(\mathbb{R})$ be real valued and let f_a be the analytic part of f. Recall that $\widehat{f_a}(\omega) = 2\widehat{f}(\omega)$ for $\omega > 0$ and $2\|f\|_2^2 = \|f_a\|_2^2$. Then:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \|f * \bar{\psi}_j\|_2^2 &= \sum_{j\in\mathbb{Z}} \int_{-\infty}^{+\infty} |f * \bar{\psi}_j(t)|^2 \, dt \\
&= \frac{1}{2\pi} \sum_{j\in\mathbb{Z}} \int_{-\infty}^{+\infty} |\widehat{f}(\omega)|^2 |\widehat{\psi}(2^j\omega)|^2 \, d\omega \\
&= \frac{1}{2\pi} \int_0^{+\infty} |\widehat{f}(\omega)|^2 \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 \, d\omega \\
&= \frac{1}{4}\frac{1}{2\pi} \int_0^{+\infty} |\widehat{f_a}(\omega)|^2 \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 \, d\omega \\
&\geq \frac{A}{2}\frac{1}{2\pi} \int_0^{+\infty} |\widehat{f_a}(\omega)|^2 \, d\omega \\
&= \frac{A}{2}\|f_a\|_2^2 \\
&= A\|f\|_2^2
\end{aligned}$$

A similar argument shows that $\sum_j \|f * \bar{\psi}_j\|_2^2 \leq B\|f\|_2^2$. Therefore the dyadic wavelet transform with a complex analytic wavelet defines a semi-discrete frame with frame bounds A and B if (65) holds.

Now suppose we only want to compute the dyadic wavelet transform up to a maximum scale 2^j for $j < J$. The lost low frequency information is captured by a single scaling function (or low pass filter). Let $\phi \in \mathbf{L}^2(\mathbb{R})$ be a low pass filter and let $\phi_J(t) = 2^{-J}\phi(2^{-J}t)$ and let ψ be a real valued wavelet. The dyadic wavelet transform in this case is defined as:

$$W_J f = \{f * \bar{\phi}_J(u), \; f * \bar{\psi}_j(u)\}_{u\in\mathbb{R}, j<J}$$

The operator W_J is the analysis operator of a semi-discrete frame if

$$A \leq |\widehat{\phi}(2^J\omega)|^2 + \sum_{j<J} |\widehat{\psi}(2^j\omega)|^2 \leq B$$

If the family $\{\psi_j\}_{j\in\mathbb{Z}}$ are the generators of a semi-discrete frame, meaning that (64) holds, then one can define ϕ in frequency as:

$$|\widehat{\phi}(\omega)|^2 = \begin{cases} (A+B)/2, & \omega = 0 \\ \sum_{j\geq 0} |\widehat{\psi}(2^j\omega)|^2, & \omega \neq 0 \end{cases}$$

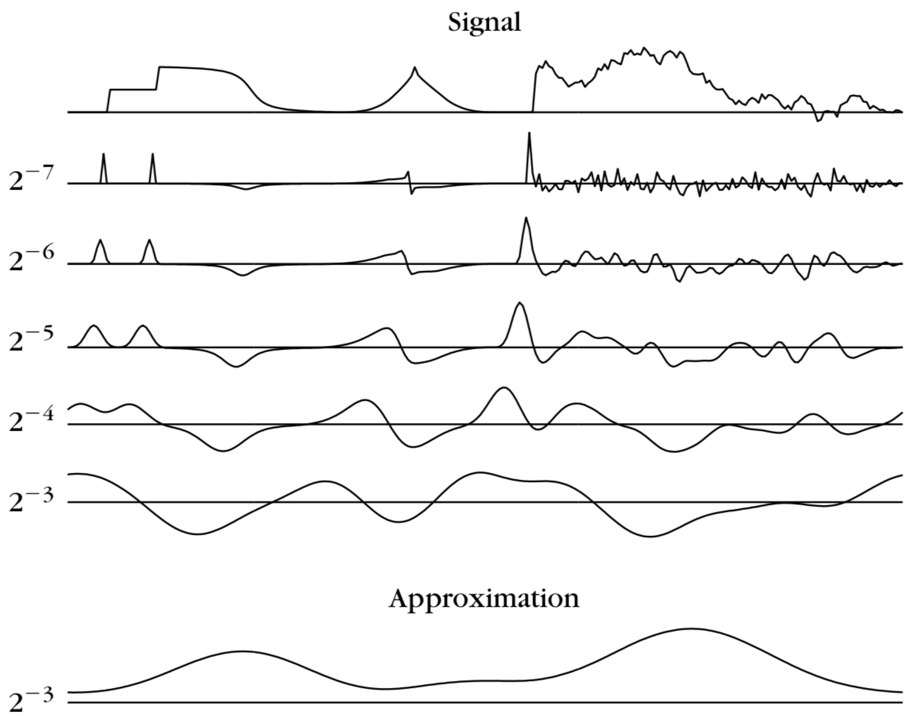

Figure 29: The dyadic wavelet transform $W_J f$ computed with $J = -2$ and $-7 \leq j \leq -3$. The top curve is $f(t)$, the next five curves are $f * \bar{\psi}_j(u)$, and the bottom curve is $f * \bar{\phi}_J$.

Figure 29 plots the dyadic wavelet transform $W_J f$ for the signal f from Figure 13.

A dual wavelet for a semi-discrete dyadic wavelet frame (without scaling function) is computed in frequency as:

$$\widehat{\widetilde{\psi}}(\omega) = \frac{\widehat{\psi}(\omega)}{\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^k\omega)|^2}$$

and the generators of the dual semi-discrete dictionary are given by the dilations of $\widetilde{\psi}$, namely $\{\widetilde{\psi}_j\}_{j\in\mathbb{Z}}$. From this definition it follows that the Fourier transform of $\widetilde{\psi}_j$ satisfies:

$$\widehat{\widetilde{\psi}}_j(\omega) = \widehat{\widetilde{\psi}}(2^j\omega) = \frac{\widehat{\psi}(2^j\omega)}{\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^{j+k}\omega)|^2} = \frac{\widehat{\psi}(2^j\omega)}{\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^k\omega)|^2}$$

We thus have

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}_j^*(\omega)\widehat{\widetilde{\psi}}_j(\omega) = \sum_{j\in\mathbb{Z}} \widehat{\psi}^*(2^j\omega)\widehat{\widetilde{\psi}}(2^j\omega) = 1, \quad \forall\, \omega \in \mathbb{R} \setminus \{0\}$$

and so by Theorem 44 the following reconstruction formula holds:

$$f(t) = \sum_{j\in\mathbb{Z}} f * \bar{\psi}_j * \widetilde{\psi}_j(t)$$

Things are simplified when the semi-discrete dyadic wavelet frame is tight. In this case

$$\widetilde{\psi}_{u,j}(t) = \frac{1}{A}\psi_{u,j}(t) = \frac{1}{A}2^{-j}\psi(2^{-j}(t-u))$$

and signal synthesis is computed as:

$$f(t) = \frac{1}{A}\sum_{j\in\mathbb{Z}} f * \bar{\psi}_j * \psi_j(t)$$

Exercise 71. Read Section 5.2 of *A Wavelet Tour of Signal Processing.*

6.2.1 Dyadic Maxima Representation

Section 6.2.2 of A Wavelet Tour of Signal Processing

Let us now temporarily return to the analysis of pointwise singularities of signals f via the decay of $Wf(u,s)$ as $s \to 0$. Let ψ be a real valued wavelet, and recall that a wavelet modulus maxima is defined as a point (u_0, s_0) such that $|Wf(u,s_0)|$ is locally maximum at $u = u_0$.

All of the results regarding wavelet coefficient decay and the pointwise regularity of $f(t)$ (including, in particular, Theorems 31, 32, and 33) hold for dyadic wavelet semi-discrete frames by restricting $s = 2^j$ for $j \in \mathbb{Z}$. Let (u_0, j) be a modulus maxima point of $Wf(u,j)$, meaning that

$$\frac{\partial Wf}{\partial u}(u_0, j) = 0 \tag{66}$$

Since $Wf(u,j) = f * \bar{\psi}_j(u)$, $\bar{\psi}_j(t) = 2^{-j}\psi(-2^{-j}t)$ and

$$\frac{d}{dt}\bar{\psi}_j(t) = -2^{-j}2^{-j}\psi'(-2^{-j}t) = -2^{-j}\overline{\psi'}_j(t)$$

equation (66) is equivalent to

$$f * \overline{\psi'}_j(u_0) = 0$$

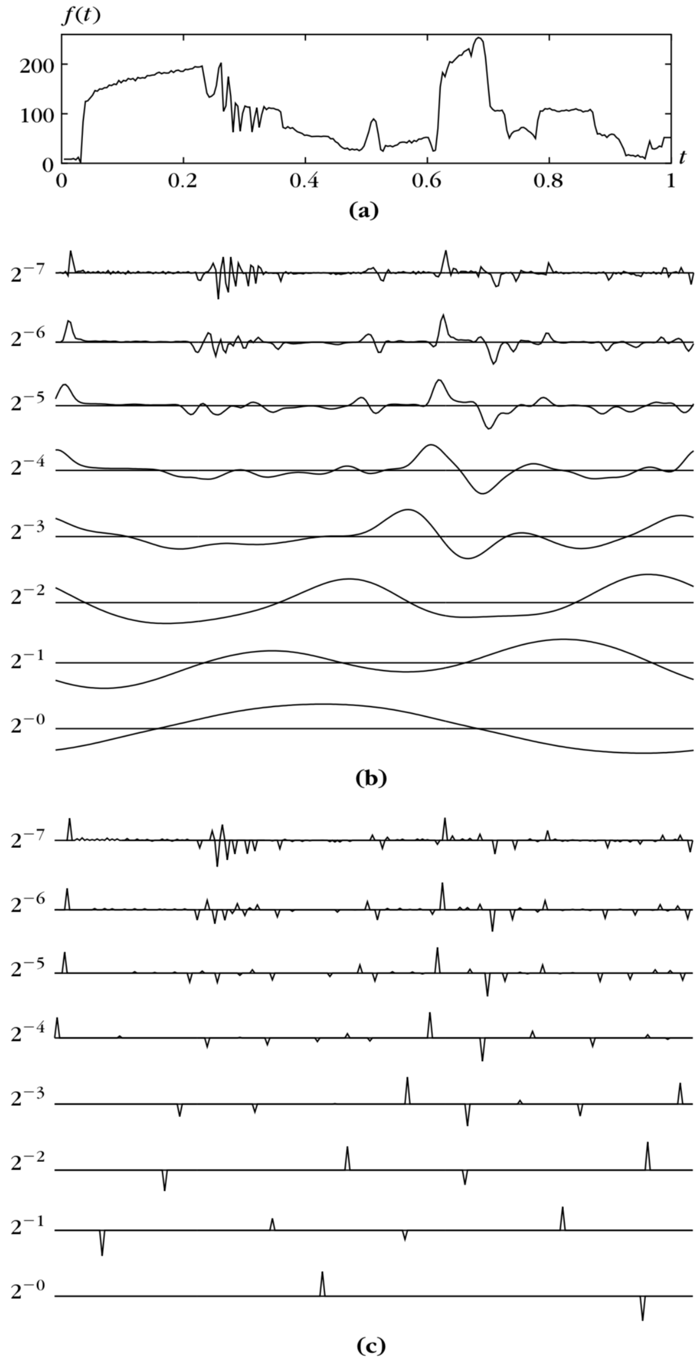

Figure 30: (a) The signal $f(t)$. (b) Dyadic wavelet transform computed with a wavelet $\psi = -\theta'$. (c) Modulus maxima of the dyadic wavelet transform.

Figure 30 shows the dyadic wavelet transform of a signal and the corresponding wavelet modulus maxima.

Let Λ denote the wavelet modulus maxima of f:

$$\Lambda = \{(u,j) \in \mathbb{R} \times \mathbb{Z} : f * \overline{\psi'}_j(u) = 0\}$$

Recall that the dictionary $\mathcal{D}$ of a dyadic wavelet transform is:

$$\mathcal{D} = \{\psi_{u,j}\}_{(u,j)\in\mathbb{R}\times\mathbb{Z}}$$

The set Λ defines a sub-dictionary of $\mathcal{D}$:

$$\mathcal{D}_\Lambda = \{\psi_{u,j}\}_{(u,j)\in\Lambda}$$

Furthermore, the completion of the span of $\mathcal{D}_\Lambda$ defines a closed subspace $\mathbf{V}_\Lambda$ of $\mathbf{L}^2(\mathbb{R})$:

$$\mathbf{V}_\Lambda = \overline{\operatorname{span} \mathcal{D}_\Lambda}$$

We can therefore project f onto $\mathbf{V}_\Lambda$. Doing so amounts to computing an approximation f_Λ of f which is the signal synthesized from only the wavelet modulus maxima of f. It is computed with a dual synthesis as:

$$f_\Lambda = P_{\mathbf{V}_\Lambda} f = \sum_{(u,j)\in\Lambda} \langle f, \psi_{u,j}\rangle \widetilde{\psi}_{u,j}$$

For general dyadic wavelets, $f_\Lambda \neq f$. However, signals with the same modulus maxima differ from each other by small amplitude errors introducing no oscillations, so in numerical experiments $f_\Lambda \approx f$. If f is band-limited (meaning it has a compactly supported Fourier transform) and ψ is as well, then the wavelet modulus maxima define a complete representation of f and in this case $f_\Lambda = f$.

Figure 31 computes the projection f_Λ for the signal first introduced in Figure 30. The signal is not bandlimited, so the reconstruction is not perfect. However, Figure 31(b) shows that the approximation is quite good, and the relative error is approximately 2.5%. Figure 31 reconstructs the signal using only the top 50% of the wavelet modulus maxima coefficients. The sharpest signal transitions have been preserved, since they have the largest amplitude responses, however small texture variations are removed since the wavelet modulus maxima there have relatively small amplitudes. The resulting signal appears to be piecewise regular.

Exercise 72. Read Section 6.2.2 of *A Wavelet Tour of Signal Processing.*

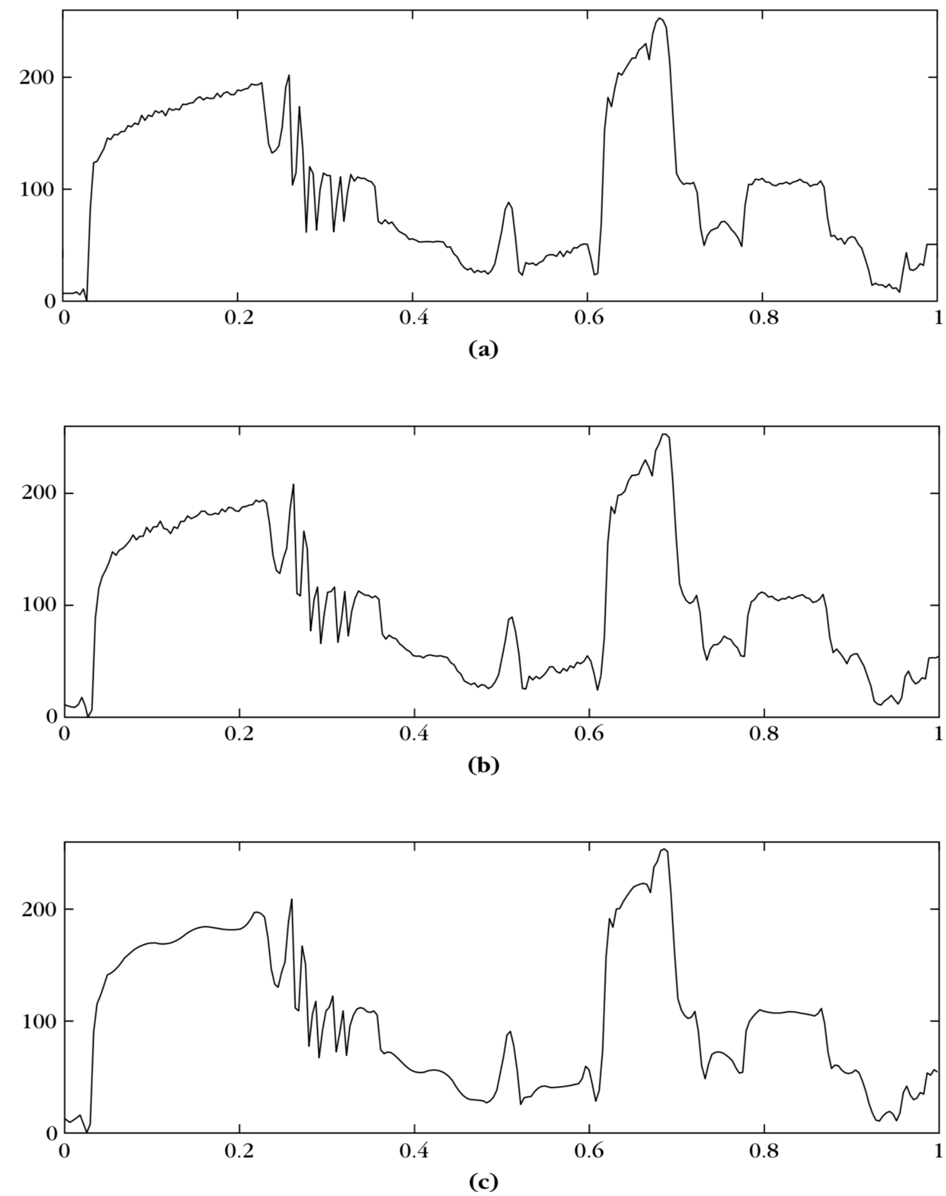

Figure 31: (a) The signal $f(t)$. (b) Signal approximation $f_\Lambda(t)$ using the dyadic wavelet modulus maxima shown in Figure 30. (c) Approximation recovered using only the largest 50% of the wavelet modulus maxima.

6.3 Subsampled Wavelet Frames

Section 5.3 of A Wavelet Tour of Signal Processing.

Wavelet frames are constructed by sampling the scale parameter s and the translation parameter u. In this section we explain how. Recall that a dyadic wavelet transform, which corresponded to semi-discrete frame, sampled the scale parameter s as $s = 2^j$ for $j \in \mathbb{Z}$. In this section we use the more general sampling $s = a^j$ for $j \in \mathbb{Z}$ with $a > 1$. Often one takes $a = 2^{1/Q}$ for $Q \in \mathbb{Z}$, $Q \geq 1$, which corresponds to putting Q wavelets in every dyadic frequency octave. The translation parameter u is sampled uniformly at intervals proportional to the scale a^j with a step size u_0:

$$\psi_{j,n}(t) = a^{-j}\psi\left(\frac{t - nu_0a^j}{a_j}\right)$$

The wavelet dictionary is then:

$$\mathcal{D} = \{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}^2}$$

In what follows we give (without proof) necessary and sufficient conditions for $\mathcal{D}$ to be a frame.

Recall that the wavelet ψ is admissible if

$$C_\psi = \int_0^{+\infty} \frac{|\widehat{\psi}(\omega)|^2}{\omega}\, d\omega < \infty$$

The next theorem is a partial analogue of Theorem 44 for wavelet frames.

Theorem 45 (Daubechies). *If $\mathcal{D} = \{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}^2}$ is a frame for $\mathbf{L}^2(\mathbb{R})$, then the frame bounds satisfy*

$$A \leq \frac{C_\psi}{u_0 \log_e a} \leq B$$

and

$$A \leq \frac{1}{u_0} \sum_{j\in\mathbb{Z}} |\widehat{\psi}(a^j\omega)|^2 \leq B$$

Note that the theorem is necessary condition for $\mathcal{D}$ to be a frame, not sufficient. We address sufficiency with the next theorem.

Theorem 46 (Daubechies). *Let us define*

$$\theta(\xi) = \sup_{1 \leq |\omega| \leq a} \sum_{j \in \mathbb{Z}} |\widehat{\psi}(a^j \omega)||\widehat{\psi}(a^j \omega + \xi)|$$

and

$$\Delta = \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \left[\theta\left(\frac{2\pi k}{u_0}\right) \theta\left(\frac{-2\pi k}{u_0}\right) \right]^{1/2}$$

If u_0 and a are such that

$$A_0 = \frac{1}{u_0} \left(\sup_{1 \leq |\omega| \leq a} \sum_{j \in \mathbb{Z}} |\widehat{\psi}(a^j \omega)|^2 - \Delta \right) > 0$$

and

$$B_0 = \frac{1}{u_0} \left(\sup_{1 \leq |\omega| \leq a} \sum_{j \in \mathbb{Z}} |\widehat{\psi}(a^j \omega)|^2 + \Delta \right) < \infty$$

then $\mathcal{D} = \{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$ is a frame for $\mathbf{L}^2(\mathbb{R})$. The frame bounds A and B satisfy $A \geq A_0$ and $B \leq B_0$.

Notice these sufficient conditions are similar to the necessary condition of Theorem 45, but do have the added Δ term. If Δ is small then A_0 and B_0 are close to the optimal frame bounds A and B. For a fixed dilation step a, the value of Δ decreases as u_0 decreases (which means we sample more translations).

Recall the Mexican hat wavelet $\psi(t) = \theta''(t)$ where $\theta(t)$ is a Gaussian function. Figure 32 gives the estimated frame bounds A_0 and B_0 computed with Theorem 46 for a wavelet frame generated with a Mexican hat wavelet for $a = 2^{1/Q}$ with $Q = 1, 2, 4$ and for various u_0. For $Q \geq 2$ the frame is nearly tight so long as $u_0 \leq 1/2$. On the other hand, for larger u_0 the ratio B_0/A_0 gets quite large, indicating that the resulting analysis and synthesis transforms will be unstable.

Exercise 73. Read Section 5.3 of *A Wavelet Tour of Signal Processing.*

Exercise 74. Read Section 5.4 of *A Wavelet Tour of Signal Processing.*

Table 5.2 Estimated Frame Bounds for the Mexican Hat Wavelet

a	u_0	A_0	B_0	B_0/A_0
2	0.25	13.091	14.183	1.083
2	0.5	6.546	7.092	1.083
2	1.0	3.223	3.596	1.116
2	1.5	0.325	4.221	12.986
$2^{\frac{1}{2}}$	0.25	27.273	27.278	1.0002
$2^{\frac{1}{2}}$	0.5	13.673	13.639	1.0002
$2^{\frac{1}{2}}$	1.0	6.768	6.870	1.015
$2^{\frac{1}{2}}$	1.75	0.517	7.276	14.061
$2^{\frac{1}{4}}$	0.25	54.552	54.552	1.0000
$2^{\frac{1}{4}}$	0.5	27.276	27.276	1.0000
$2^{\frac{1}{4}}$	1.0	13.586	13.690	1.007
$2^{\frac{1}{4}}$	1.75	2.928	12.659	4.324

Source: Computed with Theorem 5.16 [19].

Figure 32: Estimated frame bounds for the Mexican hat wavelet.

6.4 Multiscale Directional Frames for Images

Section 5.5 of A Wavelet Tour of Signal Processing.

6.4.1 Directional Wavelet Frames

Section 5.5.1 of A Wavelet Tour of Signal Processing.

We now consider two dimensional wavelet semi-discrete frames for image analysis. Such semi-discrete frames are constructed with wavelets have directional sensitivity, providing information on the direction of sharp transitions such as edges and textures.

Let $x = (x_1, x_2) \in \mathbb{R}^2$. A directional wavelet $\psi_\alpha(x)$ of angle $\alpha \in [0, 2\pi)$ is a wavelet having p directional vanishing moments along any one dimensional line of direction $\alpha + \pi/2$ in the plane but does not have directional vanishing moments along the direction α. The former condition means that:

$$\int_{-\infty}^{+\infty} \psi_\alpha(\rho \cos\alpha - u \sin\alpha, \rho \sin\alpha + u \cos\alpha) u^k \, du = 0, \quad \forall \rho \in \mathbb{R}, \, 0 \leq k < p$$

Such a wavelet oscillates in the direction $\alpha + \pi/2$ but not in the direction α.

Let $\Theta \subset [0, \pi)$ denote the set of angles α. Typically Θ is a uniform sampling:

$$\Theta = \{\alpha = 2\pi k/K : 0 \leq k < K\}$$

The generators of a translation invariant dictionary are the dyadic dilations of each directional wavelet:

$$\{\psi_{j,\alpha}\}_{j \in \mathbb{Z},\, \alpha \in \Theta}, \quad \psi_{j,\alpha}(x) = 2^{-2j}\psi_\alpha(2^{-j}x)$$

Often the directional wavelets ψ_α are obtained by rotating a single mother wavelet ψ; we will come back to this shortly when we define two dimensional Gabor and Morlet wavelets. For real valued directional wavelets, Theorem 44 proves that the generating wavelets generate a semi-discrete frame if and only if there exists $0 < A \leq B < \infty$ such that

$$A \leq \sum_{j \in \mathbb{Z}} \sum_{\alpha \in \Theta} |\widehat{\psi}_\alpha(2^j\omega)|^2 \leq B, \quad \forall\, \omega \in \mathbb{R}^2 \setminus \{(0,0)\}$$

If the generating wavelets ψ_α are complex valued analytic wavelets, then they generate a semi-discrete frame if and only if

$$2A \leq \sum_{\alpha \in \Theta} |\widehat{\psi}_\alpha(2^j\omega)|^2 + \sum_{\alpha \in \Theta} |\widehat{\psi}_\alpha(-2^j\omega)|^2 \leq 2B, \quad \forall\, \omega \in \mathbb{R}^2 \setminus \{(0,0)\} \tag{67}$$

When the above semi-discrete frame conditions holds, the dyadic directional wavelet transform is a map $W : \mathbf{L}^2(\mathbb{R}^2) \to \boldsymbol{\ell}^2(\mathbf{L}^2(\mathbb{R}^2))$ defined as:

$$Wf = \{f * \bar{\psi}_{j,\alpha}(u) : j \in \mathbb{Z},\, \alpha \in \Theta,\, u \in \mathbb{R}^2\}, \quad \bar{\psi}_{j,\alpha}(x) = \psi^*_{j,\alpha}(-x)$$

A wavelet $\psi_{u,j,\alpha}(x) = \psi_{j,\alpha}(x-u)$ has support dilated by 2^j, located in a neighborhood of u and oscillates in the direction $\alpha + \pi/2$. If $f(x)$ is constant over the support of $\psi_{j,\alpha}(x-u)$ along lines of direction $\alpha + \pi/2$, then $f * \bar{\psi}_{j,\alpha}(u) = 0$ because of its directional vanishing moments. In particular, the wavelet coefficient vanishes in the neighborhood of an edge having a tangent in the direction of $\alpha + \pi/2$. If the edge angle deviates from $\alpha + \pi/2$, then it produces large amplitude coefficients, with a maximum typically when the edge has direction α. Figure 33 illustrates the idea.

Exercise 75. Let h be a low pass filter with $\widehat{h}(0) = \sqrt{2}$ and let $\phi \in \mathbf{L}^2(\mathbb{R})$ be a scaling function with the following Fourier transform:

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2}}\widehat{h}(\omega/2)\widehat{\phi}(\omega/2)$$

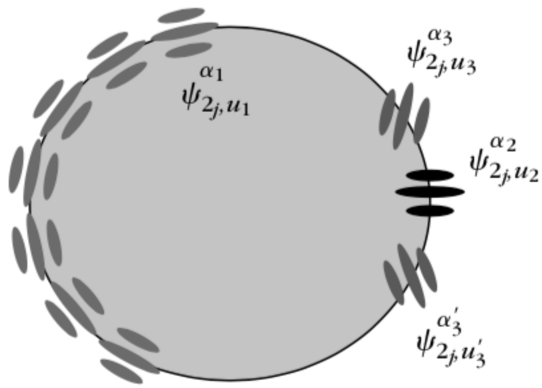

Figure 33: A cartoon image of a disk, with a regular edge. When the wavelet direction α is orthogonal to the tangent of the edge, the coefficients vanish as indicated by the black wavelet response (α_2). When the wavelet direction α aligns with the tangent of the curve (as on the left with α_1), the wavelet coefficients have large amplitude. When the tangent of the curve is not aligned with the wavelet, but is not orthogonal either (as in α_3 and α_3'), wavelet coefficients may have non-negligible amplitude but generally not as large as the α_1 coefficients.

Let g be a high pass filter with $\widehat{g}(0) = 0$ and let ψ be a wavelet with Fourier transform:

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{2}}\widehat{g}(\omega/2)\widehat{\phi}(\omega/2)$$

Prove that if there exist $0 < A \leq B < \infty$ such that

$$A(2 - |\widehat{h}(\omega)|^2) \leq |\widehat{g}(\omega)|^2 \leq B(2 - |\widehat{h}(\omega)|^2$$

then the family $\{\psi_j\}_{j\in\mathbb{Z}}$ are the generators of a semi-discrete frame.

Exercise 76. [20 points] Using your code from previous exercises compute the dyadic wavelet transform of the signal in Figure 13. Compute the wavelet modulus maxima as well. Implement a dual synthesis projection (however you like) and compute f_Λ, i.e., the signal synthesized from only the wavelet modulus maxima coefficients. Threshold the wavelet modulus maxima coefficients and synthesize a signal only from the largest ones. Turn in plots of the wavelet coefficients, the wavelet modulus maxima, and the synthesized signals. Explain your results.

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