Uncertainty Principles for Finite Abelian Groups

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Outline

- The Donoho-Stark Uncertainty Principle
 - Theory
 - Generalization to Finite Abelian Groups
 - Limiting Examples
- 2 An Uncertainty Principle for Cyclic Groups of Prime Order
 - Theory
 - Consequences

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The Fourier Transform on $\mathbb{Z}/N\mathbb{Z}$

• $l^2(\mathbb{Z}/N\mathbb{Z}) := \{ f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C} \}$

Definition

Let $f \in l^2(\mathbb{Z}/N\mathbb{Z})$. The Fourier transform of f, denoted \hat{f} , is

$$\hat{f}(\omega) := \frac{1}{\sqrt{N}} \sum_{t \in \mathbb{Z}/N\mathbb{Z}} f(t) e^{-2\pi i \omega t/N}, \qquad \omega \in \mathbb{Z}/N\mathbb{Z}$$

The Donoho-Stark Uncertainty Principle

- $supp(f) := \{t \in \mathbb{Z}/N\mathbb{Z} : f(t) \neq 0\}$
- Let $N_t = |\mathrm{supp}(f)|$ and $N_\omega = |\mathrm{supp}(\hat{f})|$

Theorem (Donoho and Stark 1989)

If $f \in l^2(\mathbb{Z}/N\mathbb{Z})$ is a non-zero function, then

$$\begin{array}{ccc} N_t N_\omega & \geq & N \\ N_t + N_\omega & \geq & 2\sqrt{N} \end{array}$$

Lemma

If $|\operatorname{supp}(f)| = N_t$, then \hat{f} cannot have N_t consecutive zeroes.

- Suppose N_t divides N
- ullet Partition $\mathbb{Z}/N\mathbb{Z}$ into N/N_t intervals of length N_t
- By the lemma, each interval contains at least one element of $\operatorname{supp}(\hat{f})$
- Thus $N_{\omega} \geq N/N_t$
- ullet Argument for when N_t does not divide N is similar

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- Let $s \in l^2(\mathbb{Z}/N\mathbb{Z})$ be a signal
- If we sample at every frequency, i.e., we know $\hat{s}(\omega)$ for all $\omega \in \mathbb{Z}/N\mathbb{Z}$, then we can reconstruct s via Fourier inversion

$$s(t) = \frac{1}{\sqrt{N}} \sum_{\omega \in \mathbb{Z}/N\mathbb{Z}} \hat{s}(\omega) e^{2\pi i \omega t/N}$$

- Suppose instead we only have knowledge of $r \in l^2(\mathbb{Z}/N\mathbb{Z})$, a bandlimited version of s, i.e. $r = P_B s$
- Assume

$$r(t) = P_B s(t) = \frac{1}{\sqrt{N}} \sum_{\omega \in B} \hat{s}(\omega) e^{2\pi i \omega t/N}$$

•

$$\hat{r}(\omega) = \begin{cases} \hat{s}(\omega) & \omega \in B \\ 0 & \text{otherwise} \end{cases}$$

• Set $N_{\omega} = |B^c|$

Theorem (Donoho and Stark 1989)

If it is known that s has only N_t non-zero elements, and if $2N_tN_\omega < N$, then s can be uniquely reconstructed from r.

Proof.

- Suppose that s_1 also generates r, i.e. $P_B s_1 = r = P_B s_1$
- Set $h := s_1 s \Longrightarrow P_B h = 0$.
- $\operatorname{supp}(s_1), \operatorname{supp}(s) \leq N_t \Longrightarrow \operatorname{supp}(h) \leq 2N_t = N_t'$
- $P_B h = 0 \Longrightarrow \operatorname{supp}(h) \subset B^c \Longrightarrow |\operatorname{supp}(h)| \le N_\omega$
- $N'_t N_\omega = 2N_t N_\omega < N \Longrightarrow h \equiv 0$



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- The restriction $2N_tN_\omega < N$ is extremely limiting.
- For example, even if $N_{\omega}=N/10$, then $N_{t}<5$ is needed.
- In practice, however, if the spike positions of a signal s are scattered at random, results showed that it is possible to recover many more spikes than $2N_tN_\omega < N$ indicates.
- In fact this turns out to be true, see research on compressed sensing.

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Preliminary Definitions

ullet Let G be a finite abelian additive group

Definition

Let $e: G \times G \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$. We say e is a nondegenerate *bi-character* of G if it has the following properties:

- $e(t+t',\omega) = e(t,\omega)e(t',\omega)$
- $e(t, \omega + \omega') = e(t, \omega)e(t, \omega')$
- For every $t \neq 0$ there exists an $\omega \in G$ such that $e(t, \omega) \neq 1$
- For every $\omega \neq 0$ there exists a $t \in G$ such that $e(t,\omega) \neq 1$

The Fourier Transform on G

- ullet Let |G| denote the cardinality of G
- $l^2(G) := \{f : G \to \mathbb{C}\}$

Definition

Let $f \in l^2(G)$. The Fourier transform of f, denoted \hat{f} , is

$$\hat{f}(\omega) := \frac{1}{\sqrt{|G|}} \sum_{t \in G} f(t) \overline{e(t, \omega)}, \qquad \omega \in G$$

An Uncertainty Principle for G

•
$$supp(f) = \{t \in G : f(t) \neq 0\}$$

Theorem (K.T. Smith 1990)

If $f \in l^2(G)$ is a non-zero function, then

$$|\operatorname{supp}(f)||\operatorname{supp}(\hat{f})| \ge |G|$$

 $|\operatorname{supp}(f)| + |\operatorname{supp}(\hat{f})| \ge 2\sqrt{|G|}$

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Limiting Examples

Example

- $f(t) = \delta_0(t) \Longrightarrow |\operatorname{supp}(f)| = 1$
- $\bullet \ \hat{f}(\omega) = \frac{1}{\sqrt{|G|}} \text{ for all } \omega \in G \Longrightarrow |\mathrm{supp}(\hat{f})| = |G|$

Example

- ullet Let H be a subgroup of G
- $f = \chi_H \Longrightarrow |\operatorname{supp}(f)| = |H|$
- It's not hard to show that $|\operatorname{supp}(\hat{f})| = |G|/|H|$
- Up to translation, modulation, and scalar multiplication, this is the only example where equality is attained.

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The Uncertainty Principle for $\mathbb{Z}/p\mathbb{Z}$

- Consider the special case when $G=\mathbb{Z}/p\mathbb{Z}$, where p is a prime number.
- Since $\mathbb{Z}/p\mathbb{Z}$ has no non-trivial subgroups, we'd hope to improve upon the D-S Uncertainty Principle.

Theorem (Biró; Meshulam; Tao 2005)

Let p be a prime number. If $f: \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ is a non-zero function, then

$$|\operatorname{supp}(f)| + |\operatorname{supp}(\hat{f})| \ge p + 1$$

Conversely, if A and B are two non-empty subsets of $\mathbb{Z}/p\mathbb{Z}$ such that $|A| + |B| \ge p + 1$, then there exists a function f such that $\operatorname{supp}(f) = A$ and $\operatorname{supp}(\hat{f}) = B$.



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The Uncertainty Principle for $\mathbb{Z}/p\mathbb{Z}$

Example

- This uncertainty principle is a vast improvement over the D-S uncertainty principle, when N is a prime number.
- Take N = p = 101
- D-S UP: $|\text{supp}(f)| + |\text{supp}(\hat{f})| \ge 2\sqrt{101} > 20$
- UP for $\mathbb{Z}/p\mathbb{Z}$: $|\operatorname{supp}(f)| + |\operatorname{supp}(\hat{f})| \ge 101 + 1 = 102$

First Lemma

Lemma

Let p be a prime number, n a positive integer, and let $P(z_1,\ldots,z_n)$ be a polynomial with integer coefficients. Suppose that we have n p^{th} roots of unity ζ_1,\ldots,ζ_n (not necessarily distinct) such that $P(\zeta_1,\ldots,\zeta_n)=0$. Then $P(1,\ldots,1)$ is a multiple of p.

Proof of First Lemma

Proof.

•
$$\zeta := e^{2\pi i/p} \Longrightarrow \zeta_j = \zeta^{k_j}, \quad 0 \le k_j < p$$

•
$$Q(z) := P(z^{k_1}, \dots, z^{k_n}) \mod z^p - 1$$

•
$$Q(\zeta) = 0$$
 and $Q(1) = P(1, ..., 1)$

- $deg(Q) \le p-1$ and Q has integer coefficients
- Q is an integer multiple of the minimal polynomial of ζ , $1+z+\cdots+z^{p-1}$



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Key Lemma

Lemma (Chebotarëv 1926)

Let p be a prime number and $1 \le n \le p$. Let t_1, \ldots, t_n be distinct elements of $\mathbb{Z}/p\mathbb{Z}$ and let $\omega_1, \ldots, \omega_n$ also be distinct elements of $\mathbb{Z}/p\mathbb{Z}$. Then the matrix $(e^{2\pi i t_j \omega_k/p})_{1 \le j,k \le n}$ has non-zero determinant.

- $\zeta_j:=e^{2\pi i t_j/p}\Longrightarrow$ we want $\det(\zeta_j^{\omega_k})_{1\leq j,k\leq n}\neq 0$
- $D(z_1, \ldots, z_n) := \det(z_i^{\omega_k})_{1 \le j,k \le n}$
- D has integer coefficients; however D(1, ..., 1) = 0
- $D(z_1, \ldots, z_n) = P(z_1, \ldots, z_n) \prod_{1 \le j < j' \le n} (z_j z_{j'})$
- \bullet P is a polynomial with integer coefficients; we will show $P(1,\dots,1)$ is not a multiple of p
- $I := (z_1 \frac{d}{dz_1})^0 (z_2 \frac{d}{dz_2})^1 \cdots (z_n \frac{d}{dz_n})^{n-1} D(z_1, \dots, z_n)|_{z_1 = \dots = z_n = 1}$
- $I = (n-1)!(n-2)! \cdots 0!P(1, \dots, 1)$
- ullet Therefore it suffices to show I is not a multiple of p.
- $I = \det(\omega_k^{j-1})_{1 \le j,k \le n} = \pm \prod_{1 < k < k' < n} (\omega_k \omega_{k'})$



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- D has integer coefficients; however $D(1, \ldots, 1) = 0$
- $D(z_1,...,z_n) = P(z_1,...,z_n) \prod_{1 \le j < j' \le n} (z_j z_{j'})$
- \bullet P is a polynomial with integer coefficients; we will show $P(1,\dots,1)$ is not a multiple of p
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Theory

Corollary to Key Lemma

Corollary

Let p be a prime number and T,Ω subsets of $\mathbb{Z}/p\mathbb{Z}$. Let $l^2(T)$ (resp. $l^2(\Omega)$) be the space of functions that are zero outside of T (resp. Ω). The restricted Fourier transform $\mathcal{F}_{T\to\Omega}: l^2(T)\to l^2(\Omega)$ is defined as

$$\mathcal{F}_{T \to \Omega} f := \hat{f}|_{\Omega} \text{ for all } f \in l^2(T)$$

If $|T| = |\Omega|$, then $\mathcal{F}_{T \to \Omega}$ is a bijection.

Proof of Theorem

Suppose |supp(f)| + |supp(f)| ≤ p
T := supp(f)
∃Ω ⊂ ℤ/pℤ, disjoint from supp(f̂) and |Ω| = |T|.
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Outline

- 1 The Donoho-Stark Uncertainty Principle
 - Theory
 - Generalization to Finite Abelian Groups
 - Limiting Examples
- 2 An Uncertainty Principle for Cyclic Groups of Prime Order
 - Theory
 - Consequences

Sparse Polynomials/Cauchy-Davenport Inequality

Proposition

Let $P(z) = \sum_{j=0}^{k} c_j z^{n_j}$ with $c_j \neq 0$ and $0 \leq n_0 < \ldots < n_k < p$. If P is restricted to the p^{th} roots of unity $\{z : z^p = 1\}$, then P can have at most k zeroes.

Theorem (Cauchy-Davenport Inequality)

Let A and B be non-empty subsets of $\mathbb{Z}/p\mathbb{Z}$ and set $A+B:=\{a+b:a\in A,b\in B\}$. Then

$$|A+B| \ge \min(|A|+|B|-1,p)$$

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Theorem (Candes, Romberg, and Tao 2006)

Suppose that the signal length N is a prime number. Let $\Omega \subset \mathbb{Z}/N\mathbb{Z}$, and let $f \in l^2(\mathbb{Z}/N\mathbb{Z})$ be a signal supported on T such that

$$|T| \le \frac{|\Omega|}{2}$$

Then f can be reconstructed uniquely from Ω and $\hat{f}|_{\Omega}$. Conversely, if Ω is not the set of all N frequencies, then there exist distinct f and g such that $|\mathrm{supp}(f)|, |\mathrm{supp}(g)| \leq |\Omega|/2 + 1$ and such that $\hat{f}|_{\Omega} = \hat{g}|_{\Omega}$.

Proof.

- ullet $|\Omega| < N \Longrightarrow$ we can find disjoint subsets T,S of Ω such that
 - $|T|, |S| \le |\Omega|/2 + 1$
 - $|T| + |S| = |\Omega| + 1$
- Let $\omega_0 \in \mathbb{Z}/N\mathbb{Z}$, $\omega_0 \notin \Omega$
- Corollary $\Longrightarrow \mathcal{F}_{T \cup S \to \Omega \cup \{\omega_0\}}$ is a bijection.
- Therefore $\exists\, h$ supported on $T\cup S$ such that $\hat{h}|_{\Omega}\equiv 0$ but $\hat{h}(\omega_0)\neq 0.$
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Example

- For signals of length N, where N is a prime number, this sparsity bound is far better than the one proposed based off the D-S uncertainty principle.
- Take N=101 and assume we sample 91 of the 101 frequencies (i.e. $|\Omega|=91$).
- D-S UP: $2|T||\Omega^c| < 101 \Longrightarrow 20|T| < 101 \Longrightarrow |T| \le 5$
- UP for $\mathbb{Z}/p\mathbb{Z}$: $|T| \leq |\Omega|/2 = 91/2 \Longrightarrow |T| \leq 45$

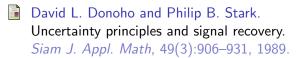


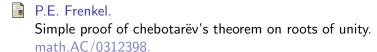
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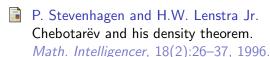


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