Minimal interpolants in $C^{1,1}(\mathbb{R}^n)$

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Overview

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Our setting

- Define the Lipschitz semi-norm of the gradient of a function $F : \mathbb{R}^n \to \mathbb{R}$ as:

$$\mathrm{Lip}(\nabla F) \triangleq \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{\|\nabla F(x) - \nabla F(y)\|}{\|x - y\|}$$

- We will work in $C^{1,1}(\mathbb{R}^n)$, which is defined as:

$$C^{1,1}(\mathbb{R}^n) \triangleq \{ F \in C^1(\mathbb{R}^n) \mid \mathrm{Lip}(\nabla F) < \infty \}$$

Input data

We are given the following data:

- The set of points that we wish to interpolate through. Denote this set as
$$E = \{p_1, \ldots, p_N\} \subset \mathbb{R}^n$$
- The function values at each point of E:
$$\begin{aligned} f : E &\to \mathbb{R} \\ p &\mapsto f(p) \end{aligned}$$
- The gradients at each point of E:
$$\begin{aligned} f_\nabla : E &\to \mathbb{R}^n \\ p &\mapsto f_\nabla(p) \end{aligned}$$

Minimal interpolant

Given (E, f, f_∇), we want to compute a minimal interpolant $F \in C^{1,1}(\mathbb{R}^n)$ such that:

- Interpolate:

$$
\begin{aligned}
F(p) &= f(p), && \text{for all } p \in E \\
\nabla F(p) &= f_\nabla(p), && \text{for all } p \in E
\end{aligned}
$$

- Minimum norm: If $\widetilde{F} \in C^{1,1}(\mathbb{R}^n)$ is another interpolant for the data (E, f, f_∇), then

$$\mathrm{Lip}(\nabla F) \leq \mathrm{Lip}(\nabla \widetilde{F})$$

Overview

1. Introduction
2. A minimal interpolant
   - An interpolant (Wells)
   - The minimal norm (Le Gruyer)
   - Le Gruyer + Wells
3. Wells' construction
   - Dual cell complexes
   - Defining the interpolant
4. Implementation on a computer
   - The model of computation
   - Efficient computation for $C^{1,1}(\mathbb{R}^2)$

Wells' theorem

- Let $M > 0$, and suppose for now that we just want an interpolant F for the data (E, f, f_∇) such that $\mathrm{Lip}(\nabla F) = M$. When is this possible?

Theorem (Wells, 1973)

Let $M > 0$ and suppose you are given data (E, f, f_∇). If for each pair $(p, p') \in E \times E$,

$$f(p') \leq f(p) + \frac{1}{2}\langle f_\nabla(p) + f_\nabla(p'), p' - p\rangle + \frac{M}{4}\|p' - p\|^2 - \frac{1}{4M}\|f_\nabla(p') - f_\nabla(p)\|^2,$$

then there exists an interpolant $F \in C^{1,1}(\mathbb{R}^n)$ for the data (E, f, f_∇) such that $\mathrm{Lip}(\nabla F) = M$. Furthermore you can explicitly construct such an F.

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Le Gruyer's Γ^1 constant

- Define $\mathrm{Lip}(E,f,f_\nabla)$ as:

$$\mathrm{Lip}(E,f,f_\nabla) \triangleq \inf\{\mathrm{Lip}(\nabla F) \mid F \text{ interpolates the data } (E,f,f_\nabla)\}$$

Definition (Le Gruyer, 2009)

Also define $\Gamma^1(E,f,f_\nabla)$ as

$$\Gamma^1(E,f,f_\nabla) \triangleq 2 \max_{\substack{p,p' \in E \\ p \neq p'}} \sup_{x \in \mathbb{R}^n} \frac{T_p(x) - T_{p'}(x)}{\|p-x\|^2 + \|p'-x\|^2},$$

where $T_p(x) \triangleq f(p) + \langle f_\nabla(p), x-p \rangle$.

Theorem (Le Gruyer, 2009)

For any data (E,f,f_∇),

$$\Gamma^1(E,f,f_\nabla) = \mathrm{Lip}(E,f,f_\nabla)$$

Overview

Combining the results of Le Gruyer and Wells

Mini-lemma

Given data (E,f,f_∇), Le Gruyer's constant $\Gamma^1 = \Gamma^1(E,f,f_\nabla)$ satisfies the condition of Wells, i.e., for each pair $(p,p') \in E \times E$,

$$f(p') \leq f(p) + \frac{1}{2}\langle f_\nabla(p) + f_\nabla(p'), p' - p\rangle + \frac{\Gamma^1}{4}\|p' - p\|^2 - \frac{1}{4\Gamma^1}\|f_\nabla(p') - f_\nabla(p)\|^2$$

Theorem (Le Gruyer & Wells)

Given data (E,f,f_∇), one can construct a minimal interpolant $F \in C^{1,1}(\mathbb{R}^n)$ for (E,f,f_∇).

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Cell complex

Definition

A cell complex in $\mathbb{R}^n$ is a set $\mathcal{K}$ of convex polyhedra (called cells) in $\mathbb{R}^n$ satisfying two conditions:

1. Every face of a cell is also a cell in $\mathcal{K}$
2. If $\sigma_1, \sigma_2 \in \mathcal{K}$ and $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2

Shift the points in E

- For each $p \in E$, define $\tilde{p} \in \mathbb{R}^n$ as

$$\tilde{p} \triangleq p - \frac{f_\nabla(p)}{\Gamma^1}$$

- Furthermore, for each $S \subseteq E$, set

$$\widetilde{S} \triangleq \{\tilde{p} \mid p \in S\}$$

Some useful functions

- For each $p \in E$, define $d_p : \mathbb{R}^n \to \mathbb{R}$ as

$$d_p(x) \triangleq f(p) - \frac{1}{2\Gamma^1} \|f_\nabla(p)\|^2 + \frac{\Gamma^1}{4} \|x - \tilde{p}\|^2$$

- Furthermore, for each $S \subseteq E$, define $d_S : \mathbb{R}^n \to \mathbb{R}$ as

$$d_S(x) \triangleq \min_{p \in S} d_p(x)$$

Special subsets of E

- For each $S \subseteq E$, define

$$S_e \triangleq \{x \in \mathbb{R}^n \mid d_p(x) = d_{p'}(x), \ \ \forall p, p' \in S\}$$

- Also for each $S \subseteq E$, define

$$S_* \triangleq \{x \in \mathbb{R}^n \mid d_p(x) = d_{p'}(x) \leq d_{p''}(x), \ \ \forall p, p' \in S, \ \ \forall p'' \in E\}$$

Definition

$$K \triangleq \{S \subset E \mid \text{for some } x \in S_*, \ \ d_S(x) < d_{E \setminus S}(x)\}$$

The cell complex $\mathcal{K}_*$

Definition

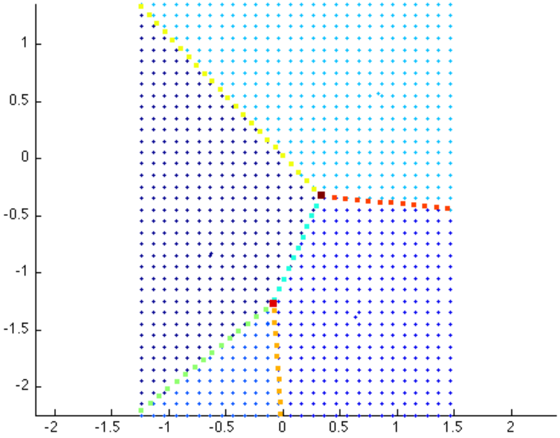
$$\mathcal{K}_* \triangleq \{S_* \mid S \in K\}$$

Lemma (Wells, 1973)

$\mathcal{K}_*$ is a cell complex in $\mathbb{R}^n$, and furthermore

$$\bigcup_{S \in K} S_* = \mathbb{R}^n$$

The cell complex $\mathcal{K}_*$



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The cell complex $\widehat{\mathcal{K}}$

- For each $S \subseteq E$, define

$$S_a \triangleq \text{the smallest affine subspace of } \mathbb{R}^n \text{ containing } \widetilde{S}$$

- Also for each $S \subseteq E$, define

$$\widehat{S} \triangleq \text{convex hull of } \widetilde{S}$$

Definition

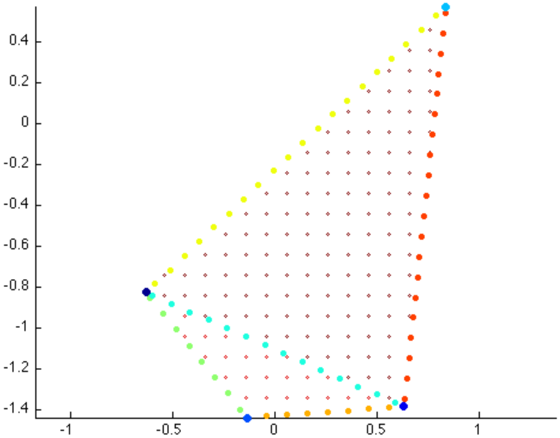
$$\widehat{\mathcal{K}} \triangleq \{\widehat{S} \mid S \in K\}$$

Lemma (Wells, 1973)

$\widehat{\mathcal{K}}$ is a cell complex in $\mathbb{R}^n$, and furthermore

$$\bigcup_{S \in K} \widehat{S} = \text{convex hull of } \widetilde{E}$$

The cell complex $\widehat{\mathcal{K}}$



Duality between $\widehat{\mathcal{K}}$ and $\mathcal{K}_*$

The two cell complexes $\widehat{\mathcal{K}}$ and $\mathcal{K}_*$ have the following duality between them:

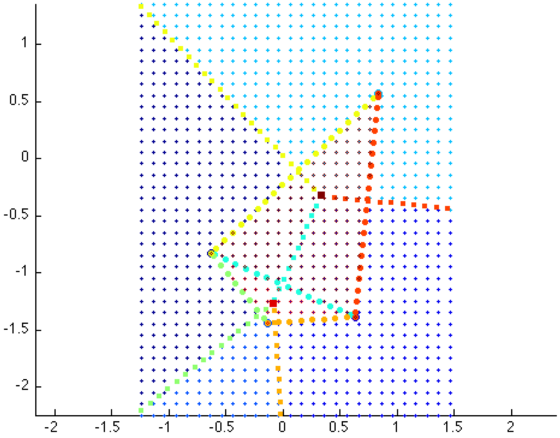
- For each $S \in K$,

$$\widehat{S} \perp S_* \quad \text{and} \quad \dim(\widehat{S}) + \dim(S_*) = n$$

- If $S, S' \in K$ and $S \subseteq S'$, then

$$\widehat{S} \subseteq \widehat{S'} \quad \text{and} \quad S'_* \subseteq S_*$$

The cell complexes $\widehat{\mathcal{K}}$ and $\mathcal{K}_*$



A third cell complex

Definition

For each $S \in K$, let

$$T_S \triangleq \{x \in R^n \mid x = \frac{1}{2}(y+z) \text{ for some } y \in \widehat{S} \text{ and } z \in S_*\},$$

and set

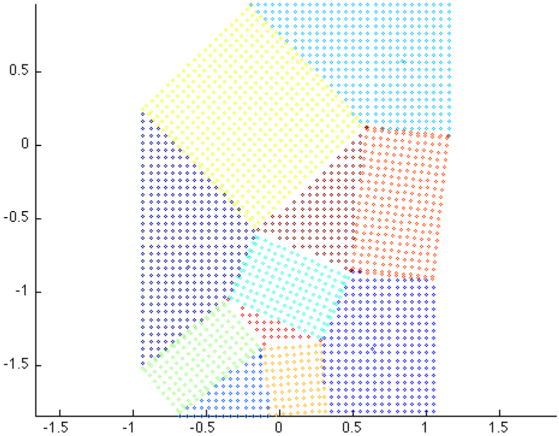
$$\mathcal{K}_T \triangleq \{T_S \mid S \in K\}$$

Lemma (Wells, 1973)

$\mathcal{K}_T$ is a cell complex in $\mathbb{R}^n$, and furthermore

$$\bigcup_{S \in K} T_S = \mathbb{R}^n$$

The cell complex $\mathcal{K}_T$



Overview

The minimal interpolant

- For each $S \in K$, set $S_c \triangleq S_a \cap S_e \in \mathbb{R}^n$ (note: $S_a \perp S_e$).

Definition

For each $S \in K$, define $F_S : T_S \to \mathbb{R}$ as

$$F_S(x) \triangleq d_S(S_c) + \frac{\Gamma^1}{2}(\mathrm{dist}(x, S_a)^2 - \mathrm{dist}(x, S_e)^2), \quad \forall x \in T_S$$

Theorem (Wells and Le Gruyer)

A minimal interpolant $F : \mathbb{R}^n \to \mathbb{R}$ can then be defined as:

$$F(x) \triangleq F_S(x), \quad \text{if } x \in T_S$$

Proof of theorem

Lemma (Wells, 1973)

For each $S \in K$, $F_S \in C^\infty(T_S)$ and furthermore,

$$\nabla F_S(x) = \frac{\Gamma^1}{2}(z - y),$$

where $x = \frac{1}{2}(y + z)$ and $y \in \widehat{S}$, $z \in S_*$.

Proof of theorem

Now one needs to check several things:

- Is F well defined?
  $F_S(x) = F_{S'}(x)$, for all $x \in T_S \cap T_{S'}$
  Proof: use $S_a \perp S_e$, $\widehat{S} \perp S_*$, and $S_c = S_a \cap S_e$
- Is $F \in C^{1,1}(\mathbb{R}^n)$?
  $\nabla F_S(x) = \nabla F_{S'}(x)$, for all $x \in T_S \cap T_{S'}$
  Proof: use the previous lemma
- Does $\mathrm{Lip}(\nabla F) = \Gamma^1$?
  $\mathrm{Lip}(\nabla F_S) = \Gamma^1$
  Proof: use previous lemma and $\widehat{S} \perp S_*$

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Proof of theorem

Finally, we need to check that F interpolates the data (E, f, f_∇):

$$F(p) = f(p) \quad \text{and} \quad \nabla F(p) = f_\nabla(p), \;\; \forall\, p \in E$$

Proof (fix $p \in E$):

- Clearly, $\{p\}_a = \widehat{\{p\}} = \{\tilde{p}\} = \{p - \frac{f_\nabla(p)}{\Gamma^1}\} \Rightarrow \{p\}_c = \tilde{p}$
- Wells' condition $\Rightarrow d_p(p + \frac{f_\nabla(p)}{\Gamma^1}) \leq d_{p'}(p + \frac{f_\nabla(p)}{\Gamma^1}), \;\; \forall\, p' \in E$
- $\Rightarrow p + \frac{f_\nabla(p)}{\Gamma^1} \in \{p\}_*$
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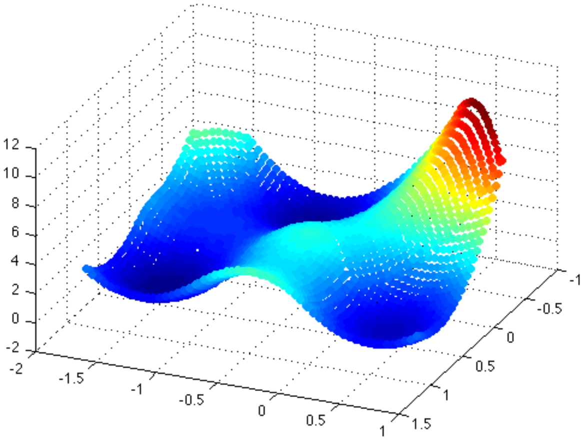
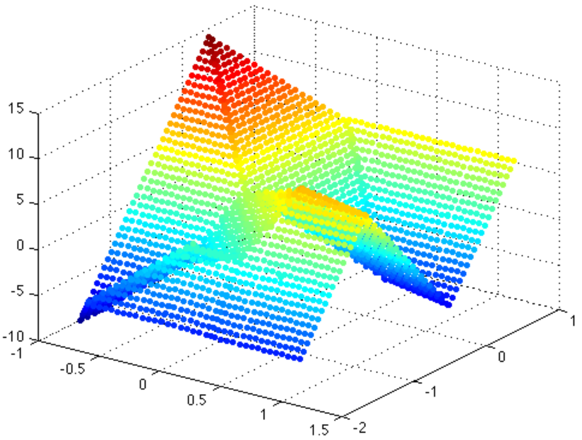
The minimal interpolant F



The partial derivative $\frac{\partial F}{\partial y}$



Overview

Computer model

We want to efficiently compute minimal interpolants $F \in C^{1,1}(\mathbb{R}^n)$ for the data (E, f, f_∇).

Our computer model is the following:

- It can work with real numbers and an exact real number can be stored at each memory address
- It takes one machine operation to add, subtract, multiply, or divide two real numbers x and y, or to compare them (i.e., decide whether $x < y$, $x > y$, or $x = y$)

Compute a function F

To "compute a function F" means the following:

- First, we enter the data (E, f, f_∇) into the computer.
- The computer then works for a while, performing L_0 machine operations. This is the one-time work.
- The computer then signals that it is ready to accept further input.
- Whenever we enter a point $x \in \mathbb{R}^n$, the computer responds by producing $(F(x), \nabla F(x))$, using L_1 machine operations to the make the computation. This is the query work.
- The storage of our algorithm is the number of memory addresses required to carry out the above work.

Efficiency goals

Recall, $|E| = N$.

Ideally, our algorithm will only require the following:

- One-time work: $C(n)NlogN$
- Query work: $C(n)logN$
- Storage: $C(n)N$

Here, $C(n)$ is a constant depending only on n.

Overview

Computation of Γ^1

- To compute a minimal interpolant F *exactly*, using only $C(n)NlogN$ one-time work, is most likely impossible.
- Indeed, recall the constant Γ^1, defined as:

$$\Gamma^1 \triangleq 2 \max_{\substack{p,p' \in E \\ p \neq p'}} \sup_{x \in \mathbb{R}^n} \frac{T_p(x) - T_{p'}(x)}{\|p-x\|^2 + \|p'-x\|^2}$$

- In fact, by a result of Le Gruyer (2009), one can write Γ^1 as:

$$\Gamma^1 = \max_{\substack{p,p' \in E \\ p \neq p'}} \sqrt{A_{p,p'}^2 + B_{p,p'}^2} + |A_{p,p'}|$$

- Still though, to compute Γ^1, we must use $C(n)N^2$ operations.

Approximation of Γ^1

- We propose computing an approximation $\widetilde{\Gamma}^1$ that is within $\varepsilon > 0$ of Γ^1, i.e.,

$$\widetilde{\Gamma}^1 \leq (1+\varepsilon)\Gamma^1$$

Conjecture/Lemma (H. and Narayanan, 2011)

Let $\varepsilon > 0$ and suppose you are given data (E, f, f_∇), where $|E| = N$. Then there exists an algorithm, using no more than $C(n,\varepsilon)N$ storage and $C(n,\varepsilon)NlogN$ work, that computes a constant $\widetilde{\Gamma}^1(E, f, f_\nabla)$ such that

$$\widetilde{\Gamma}^1 \leq (1+\varepsilon)\Gamma^1$$

- Main idea: use a well separated pairs decomposition (WSPD) for E in the same spirit as how one approximates $\mathrm{Lip}(f)$. By Callahan and Kosaraju (1995), the WSPD can be computed using only $C(n,\varepsilon)N$ storage and $C(n,\varepsilon)NlogN$ work.

Computation of F

Conjecture/Theorem (H. and Narayanan, 2011)

Let $\varepsilon > 0$ and suppose you are given data (E, f, f_∇), where $E \subset \mathbb{R}^2$ and $|E| = N$. Then there exists an algorithm, using no more than $C(\varepsilon)N$ storage, $C(\varepsilon)NlogN$ one-time work, and $C(\varepsilon)logN$ query work, that computes an interpolant F for the data (E, f, f_∇) such that $\mathrm{Lip}(\nabla F) \leq (1+\varepsilon)\Gamma^1$.

The main ideas are the following:

- Use the previous lemma to compute $\widetilde{\Gamma}^1$ (one-time work).
- The cell complex $\mathcal{K}_*$ is a generalization of a Voronoi diagram, which in $\mathbb{R}^2$ can be computed using CN storage and $CNlogN$ operations (one-time work).
- Given $x \in \mathbb{R}^2$, determining which T_S cell contains x is a point location problem. In $\mathbb{R}^2$, this can be done using CN storage and $logN$ operations (query work).

Future directions

Some possible future directions are:

- Computation of interpolants in higher dimensions.
- Consider the same space $C^{1,1}(\mathbb{R}^n)$, but suppose you are only given data (E, f) (i.e. no information on the gradients).
- Consider higher order derivatives, i.e., $C^{m,1}(\mathbb{R}^n)$.

Merci!

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