Minimal $C^{1,1}$ Extensions

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Collaborators

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Outline

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The Lipschitz extension problem

$C^{0,1}$ extension problem

What we are given:

1. A closed set $E \subset \mathbb{R}^d$.
2. A function $f : E \to \mathbb{R}$.

If possible, we want a function $F \in C^{0,1}(\mathbb{R}^d)$ such that:

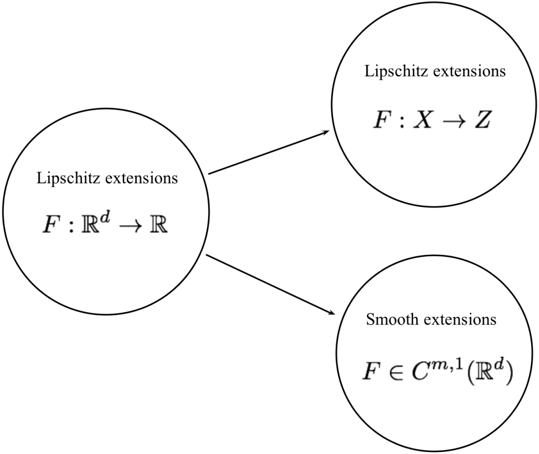
1. Extension: $F(a) = f(a)$ for each $a \in E$.
2. Minimal: For all other extensions $G \in C^{0,1}(\mathbb{R}^d)$,

$$\mathrm{Lip}(F) \leq \mathrm{Lip}(G)$$

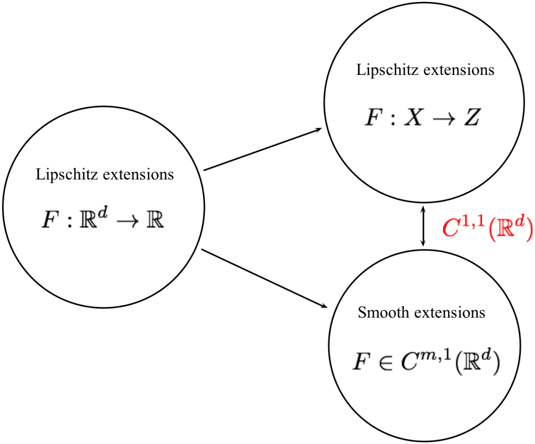
Theorem (Whitney; McShane; Kirszbraun; 1934)

If $\mathrm{Lip}(f) < \infty$, then f can be extended and furthermore, there exists an extension F such that $\mathrm{Lip}(F) = \mathrm{Lip}(f)$.

Lipschitz extension vs smooth extension



Lipschitz extension vs smooth extension



The $C^{1,1}(\mathbb{R}^d)$ extension problem

$C^{1,1}$ extension problem

What we are given:

1. A closed set $E \subset \mathbb{R}^d$.
2. A polynomial $P_a \in \mathcal{P}^1$ for each $a \in E$. We call the map

$$\begin{aligned} P : E &\to \mathcal{P}^1 \\ a &\mapsto P_a \end{aligned}$$

   a *1-field*.

If possible, we want a function $F \in C^{1,1}(\mathbb{R}^d)$ such that:

1. <u>Extension:</u> $J_aF(x) \triangleq F(a) + \nabla F(a) \cdot (x - a) = P_a(x)$ for each $a \in E$.
2. <u>Minimal:</u> For all other extensions $G \in C^{1,1}(\mathbb{R}^d)$,

$$\mathrm{Lip}(\nabla F) \leq \mathrm{Lip}(\nabla G)$$

Whitney's Extension Theorem

Theorem (Whitney's Extension Theorem for $C^{1,1}(\mathbb{R}^d)$, 1934)

Let $E \subset \mathbb{R}^d$ be a closed set with associated 1-field $P : E \to \mathcal{P}^1$. If there exists a constant M such that

1. $|P_a(a) - P_b(a)| \leq M|a-b|^2, \quad \forall\, a, b \in E$
2. $|\frac{\partial P_a}{\partial x_i}(a) - \frac{\partial P_b}{\partial x_i}(a)| \leq M|a-b|, \quad \forall\, a, b \in E, \quad i = 1, \ldots, d$

then there exists a function $F \in C^{1,1}(\mathbb{R}^d)$ that extends the 1-field, i.e.,

$$J_aF = P_a, \quad \forall\, a \in E$$

Remark: The theorem does not give the minimal value of $\mathrm{Lip}(\nabla F)$, even if one were to consider the infimum over all M satisfying the above conditions.

The minimal value of $\mathrm{Lip}(\nabla F)$

Definition

For $E \subset \mathbb{R}^d$ with 1-field $P : E \to \mathcal{P}^1$ define:

$$A(a,b) \triangleq \frac{|P_a(a) - P_b(a) + P_a(b) - P_b(b)|}{|a-b|^2}, \quad B(a,b) \triangleq \frac{|\nabla P_a - \nabla P_b|}{|a-b|}$$

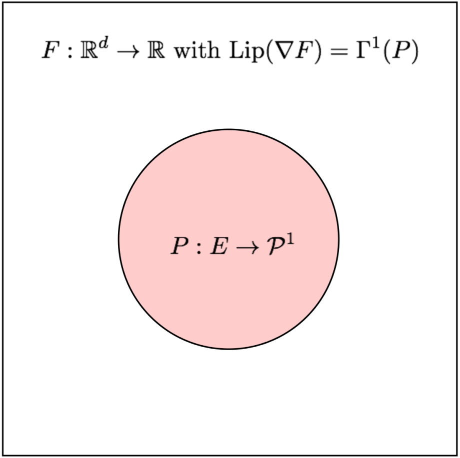
$$\Gamma^1(P) \triangleq \sup_{\substack{a,b \in E \\ a \neq b}} \sqrt{A(a,b)^2 + B(a,b)^2} + A(a,b)$$

Theorem (Le Gruyer, 2009)

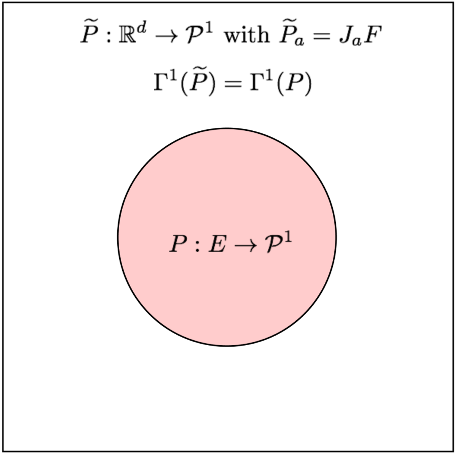
If $\Gamma^1(P) < \infty$, then there exists an extension $F \in C^{1,1}(\mathbb{R}^d)$ such that $\mathrm{Lip}(\nabla F) = \Gamma^1(P)$. Furthermore,

$$\Gamma^1(P) = \inf\{\mathrm{Lip}(\nabla G) \mid J_a G = P_a \;\; \forall\, a \in E\}$$

Lipschitz constant for 1-fields



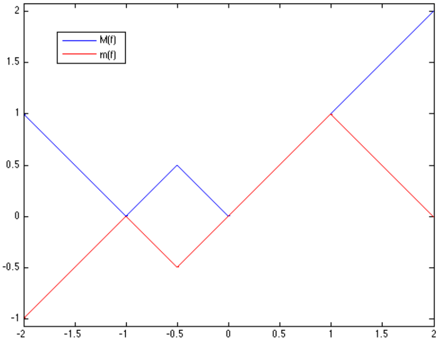
Lipschitz constant for 1-fields



Outline

Non-uniqueness of isometric Lipschitz extensions

- $f : E \to \mathbb{R}$
- $E = \{-1, 0, 1\} \subset \mathbb{R}$
- $f(-1) = f(0) = 0, f(1) = 1$



Absolutely minimal Lipschitz extensions

- $(X, d_X) =$ domain.
- $(Z, d_Z) =$ range.

Absolutely minimal Lipschitz extension (Aronsson, 1967)

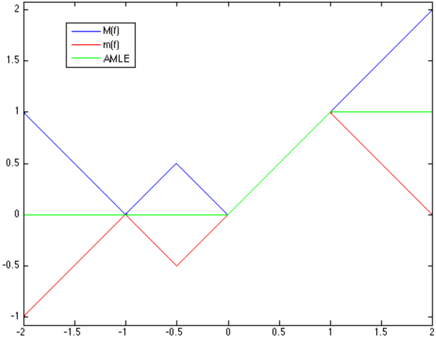
For $f : E \to Z$, let $F : X \to Z$ be an isometric extension, i.e., $\mathrm{Lip}(F) = \mathrm{Lip}(f)$. The function F is an *absolutely minimal Lipschitz extension (AMLE)* if for every open subset $V \subset X \setminus E$,

$$\mathrm{Lip}(F|_V) = \mathrm{Lip}(F|_{\partial V})$$

- The AMLE is the locally best Lipschitz extension of f.
- Again we can extend the definition to functionals Φ other than Lip, including Γ^1.

AMLE Example

- $f : E \to \mathbb{R}$
- $E = \{-1, 0, 1\} \subset \mathbb{R}$
- $f(-1) = f(0) = 0, f(1) = 1$



Existence and uniqueness for $Z = \mathbb{R}$

Existence of AMLEs when $Z = \mathbb{R}$:

- $X = \mathbb{R}^d$: Aronsson, 1967
- $(X, d_X) =$ length space: Mil'man, 1999

Uniqueness of AMLEs when $Z = \mathbb{R}$:

- $X = \mathbb{R}^d$: Jensen, 1993
- $(X, d_X) =$ length space: Peres, Schramm, Sheffield, Wilson, 2009

Relationship to PDEs

- The infinity Laplacian is defined as:

$$\Delta_\infty g \triangleq \sum_{i,j=1}^{d} \frac{\partial^2 g}{\partial x_i \partial x_j} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j}$$

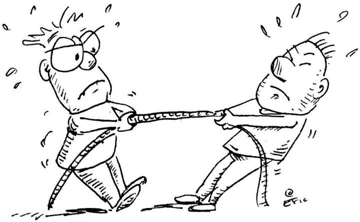
Theorem (Aronsson, 1967; Jensen, 1993)

Given $f : E \to \mathbb{R}$ with $\mathrm{Lip}(f) < \infty$, let $F : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz extension of f with $\mathrm{Lip}(F) = \mathrm{Lip}(f)$.

- *If $F \in C^2(\mathbb{R}^d)$, then F is the AMLE for $f \iff \Delta_\infty F = 0$ on $\mathbb{R}^d \setminus E$ (Aronsson).*
- *If $F \notin C^2(\mathbb{R}^d)$, then F is the AMLE for $f \iff \Delta_\infty F = 0$ on $\mathbb{R}^d \setminus E$, interpreted as a viscosity solution (Jensen).*

Tug of war

- Two player, zero-sum, random turn game.

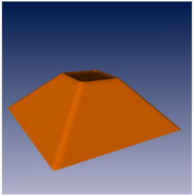


Theorem (Peres, Schramm, Sheffield, Wilson, 2009)

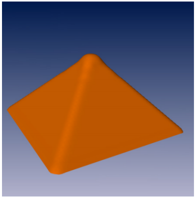
The expected payoff optimized over all possible strategies is an AMLE.

Other applications of AMLEs

- Image and surface inpainting/interpolation (various)



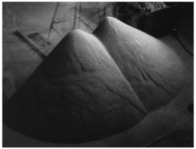
(d) Pyramid missing top



(e) Inpainted pyramid

Figure : Caselles, Haro, Sapiro, Verdera

- Analysis of shapes of sandpiles (Aronsson, Evans, Wu, 1996)



Results for $Z \neq \mathbb{R}$

Theorem (Naor and Sheffield, 2012)

AMLEs exist and are unique when

- (X, d_X) *is a locally compact length space.*
- (Z, d_Z) *is a metric tree.*

Theorem (Sheffield and Smart, 2012)

Tight AMLEs exist and are unique when:

- (X, d_X) *is a finite graph.*
- $(Z, d_Z) = \mathbb{R}^n$.

Also addresses $(X, d_X) = \mathbb{R}^d$ *and its complications.*

Quasi-AMLEs

Theorem (H. and Le Gruyer, 2014)

- $X = \mathbb{R}^d$
- $Z = \mathcal{P}^1$
- Γ^1 *which satisfies the isometric extension property*

Let $E \subset \mathbb{R}^d$ *and* $P : E \to \mathcal{P}^1$*. Then there exists a quasi-AMLE* $\widetilde{P} : \mathbb{R}^d \to \mathcal{P}^1$ *such that* $\widetilde{P}_a = P_a$ *for all* $a \in E$, $\Gamma^1(\widetilde{P}) = \Gamma^1(P)$, *and*

$$\Gamma^1(\widetilde{P}|_V) = \Gamma^1(\widetilde{P}|_{\partial V}), \quad \forall \text{ open } V \subset \mathbb{R}^d \setminus E$$

"almost" holds.

Quasi-AMLEs

Theorem (H. and Le Gruyer, 2014)

- $X = \mathbb{R}^d$
- $Z = \mathcal{P}^1$
- Γ^1 *which satisfies the isometric extension property*

Let $E \subset \mathbb{R}^d$ and $P : E \to \mathcal{P}^1$. Then there exists a quasi-AMLE $\widetilde{P} : \mathbb{R}^d \to \mathcal{P}^1$ such that $\widetilde{P}_a = P_a$ for all $a \in E$, $\Gamma^1(\widetilde{P}) = \Gamma^1(P)$, and

$$|\Gamma^1(\widetilde{P}|_V) - \Gamma^1(\widetilde{P}|_{\partial V})| < \varepsilon, \quad \forall \text{ open } V \subset X \setminus E$$

"almost" holds for all $\varepsilon > 0$.

Quasi-AMLEs

Theorem (H. and Le Gruyer, 2014)

- $X =$ *metric space with some restrictions*
- $Z =$ *complete metric space*
- Φ *which satisfies the isometric extension property*

Let $E \subset X$ *and* $f : E \to Z$. *Then there exists a quasi-AMLE* $F : X \to Z$ *such that* $F(a) = f(a)$ *for all* $a \in E$, $\Phi(F) = \Phi(f)$, *and*

$$|\Phi(F|_V) - \Phi(F|_{\partial V})| < \varepsilon \quad \forall \text{ open } V \subset X \setminus E$$

"almost" holds for all $\varepsilon > 0$.

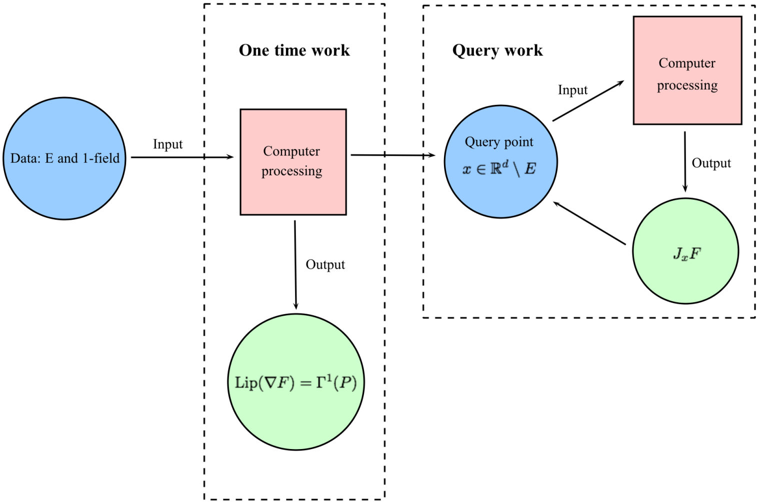
Outline

The $C^{1,1}$ Extension Problem

Absolutely Minimal Lipschitz Extensions (with E. Le Gruyer)

Algorithms (with A. Herbert-Voss and F. McCollum)

Overview of algorithm



Related work

Algorithms for $C^m(\mathbb{R}^d)$:

- Fefferman and Klartag, 2009: Computes an extension $F \in C^m(\mathbb{R}^d)$ such that

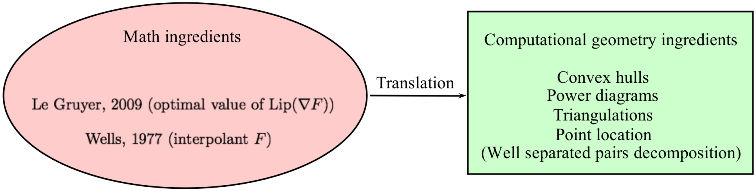
$$c(m,d)M \leq \|F\|_{C^m(\mathbb{R}^d)} \leq C(m,d)M$$

  where M is the minimum possible norm. Requires $O(N \log N)$ one time work, $O(\log N)$ query work, and $O(N)$ storage.
- Fefferman, 2010: Computes an extension $F \in C^m(\mathbb{R}^d)$ such that

$$M \leq \|F\|_{C^m(\mathbb{R}^d)} \leq (1+\varepsilon)M$$

  Requires solving a linear programming problem of size $C(m,d)|\log \eta|\varepsilon^{-\frac{3}{2}d}N$.

Ingredients of the algorithm



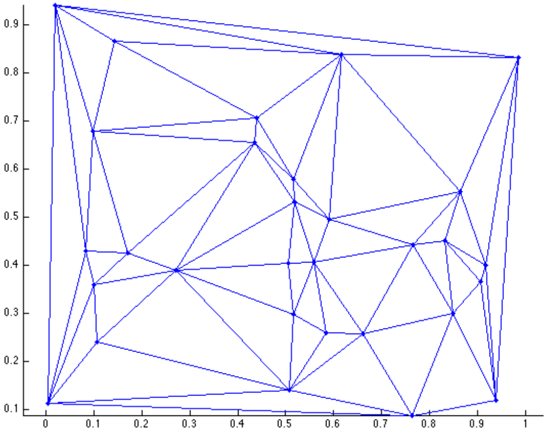
Algorithm key points

- Can compute an interpolant with the optimal value of $\mathrm{Lip}(\nabla F) = \Gamma^1(P)$.
- The interpolant is an AMLE.
- Not simple, but simple enough to implement on a computer and run on your laptop.
- Main bottleneck is computing convex hulls.
  - Worst case: $O(N^{d/2} + N \log N)$.
  - Output sensitive: $O(N^2 + K \log K)$ (Seidel, 1986).
  - Efficient output sensitive algorithms for $d \leq 5$ (Chan, Snoeyink, and Yap, 1995; Amato and Ramos, 1996).
  - Complexity of convex hull in expectation grows as $O(\log^{(d-1)(d/2)} N)$ (Dwyer, 1988).
  - Quickhull.

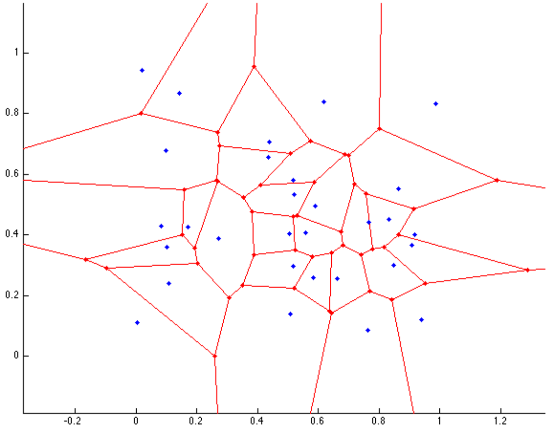
Lift points

Lifted convex hull

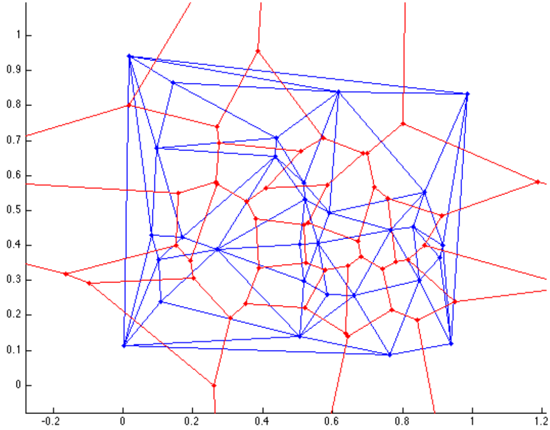
Triangulation



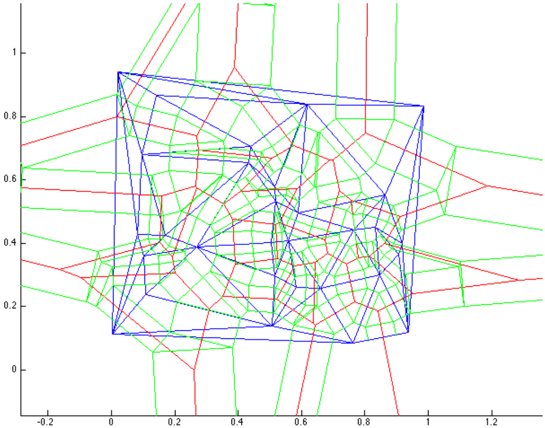
Power diagram



Power diagram with dual triangulation



Merging of the two



Extension

Open questions

$C^{m,1}(\mathbb{R}^d)$

- Can we find formulas for Γ^m?

AMLE

- Quasi-AMLE $\to$ AMLE?
- For 1-fields:
  - Correct formulation?
  - Relationship to PDEs?
  - Relationship to stochastic games?

Algorithm

- Applications?

Thank you!

`www.di.ens.fr/~hirn`

Code available at:
`http://csce.uark.edu/~fmccollu`