Frame potential classification algorithm for retinal data

Matthew Hirn

Norbert Wiener Center
Department of Mathematics
University of Maryland, College Park

RIT on Multispectral Retinal Imaging and Mapping of Naturally Occurring Fluorophore and Chromophore Distributions

University of Maryland, College Park

May 11, 2009

Norbert Wiener Center for Harmonic Analysis and Applications

Collaborators

Norbert Wiener Center:

John J. Benedetto, Wojciech Czaja, J. Christopher Flake, Nathaniel Strawn, Jesse Sugar-Moore

National Institutes of Health:

Robert Bonner, Martin Ehler, Zigurtas Majumdar

Norbert Wiener Center
for Harmonic Analysis and Applications

Outline

1. Introduction
   - Drusen
   - Multispectral imagery data
2. The algorithm
   - Overview
   - Kernel eigenmap methods
   - Frames
3. Results

Norbert Wiener Center
for Harmonic Analysis and Applications

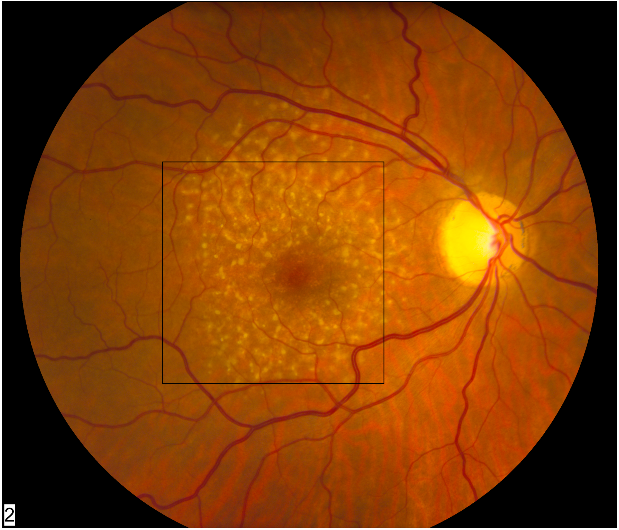
Outline

1. Introduction
   - Drusen
   - Multispectral imagery data
2. The algorithm
   - Overview
   - Kernel eigenmap methods
   - Frames
3. Results

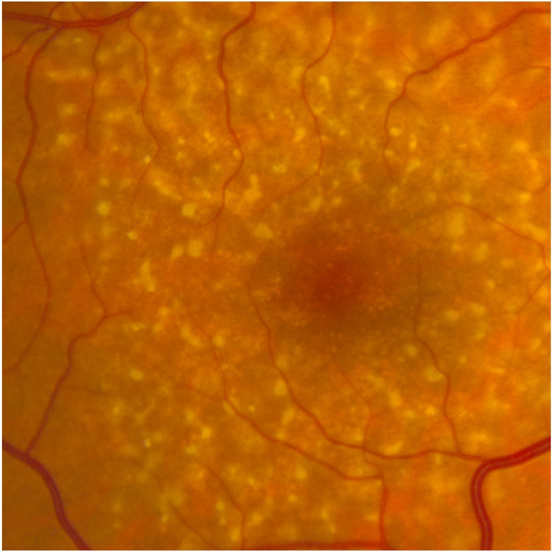
The setup

- **The problem:** Age related macular degeneration (AMD)
- **What it does:** Leading cause of blindness in elderly population
- **Indicator:** Drusen (accumulated photoproducts)
- **Goal:** Detect/classify drusen in their early stages

Color fundus image (shows drusen)



Enlarged color fundus image

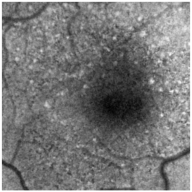


Our solution

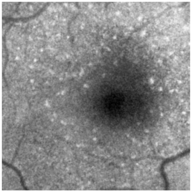
We propose the following solution for finding drusen:

- Use multispectral imagery data sets of the eye to obtain enhanced spectral information of the materials in the eye.
- Process the multispectral data set with our kernel / modified frame potential algorithm.

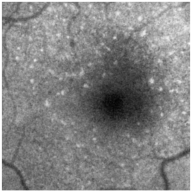
Multispectral bands



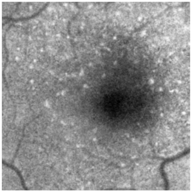
(a) Band 1



(b) Band 2



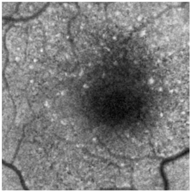
(c) Band 3



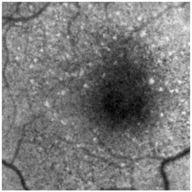
(d) Band 4

Norbert Wiener Center for Harmonic Analysis and Applications

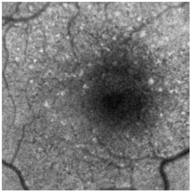
Multispectral bands



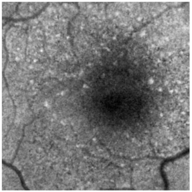
(e) Band 5



(f) Band 6

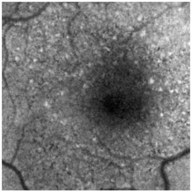


(g) Band 7

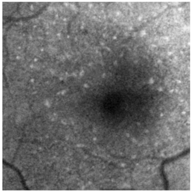


(h) Band 8

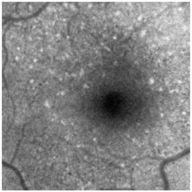
Multispectral bands



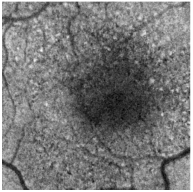
(i) Band 9



(j) Band 10



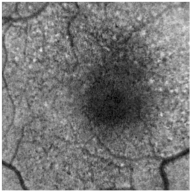
(k) Band 11



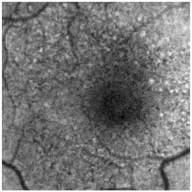
(l) Band 12

Norbert Wiener Center for Harmonic Analysis and Applications

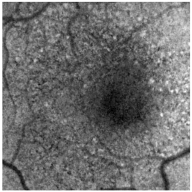
Multispectral bands



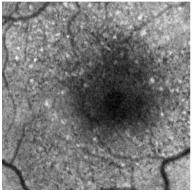
(m) Band 13



(n) Band 14



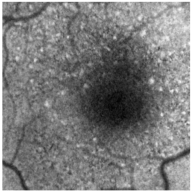
(o) Band 15



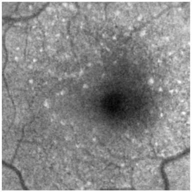
(p) Band 16

Norbert Wiener Center
for Harmonic Analysis and Applications

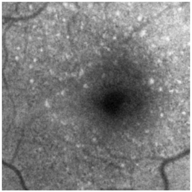
Multispectral bands



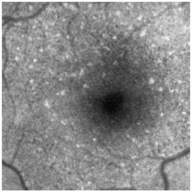
(q) Band 17



(r) Band 18

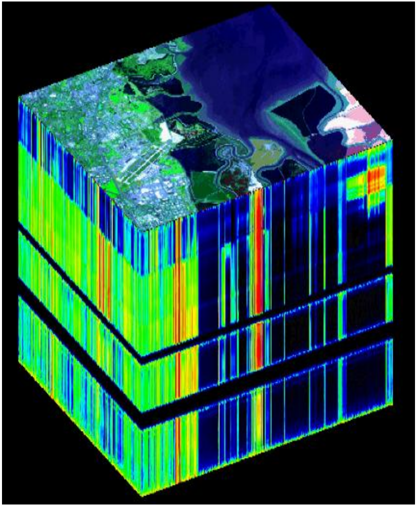


(s) Band 19



(t) Band 20

Multispectral data cube



Outline

1. Introduction
   - Drusen
   - Multispectral imagery data
2. The algorithm
   - Overview
   - Kernel eigenmap methods
   - Frames
3. Results

Notation

- Assume our multispectral data set is an $N_1 \times N_2 \times D$ cube.
- N_1, N_2 spatial dimensions; $N = N_1 N_2$ pixels.
- D is the spectral dimension; so D wavelengths measured.
- Let $X = \{x_i\}_{i=1}^N \subset \mathbb{R}^D$ denote the pixel vectors of the data cube in list form.

Introduction to the algorithm

We have two goals:

1. Map the high dimensional data set X to an appropriate low dimensional space Y.
2. Represent the low dimensional space Y for the purposes of material classification, and in particular, finding drusen.

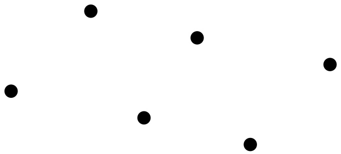
We achieve these goals via two existing mathematical theories:

1. We use kernel eigenmap methods to determine the space Y.
2. We represent Y with a data dependent frame Φ.

Norbert Wiener Center for Harmonic Analysis and Applications

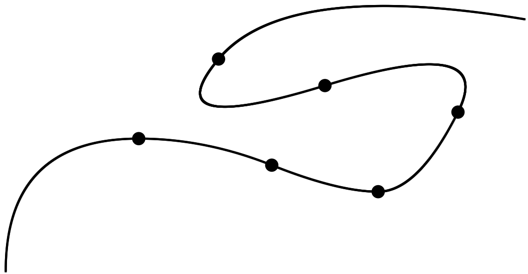
Introduction to kernel eigenmap methods

- Given a high dimensional data set $X = \{x_i\}_{i=1}^{N} \subset \mathbb{R}^D$, we assume X lies on a low dimensional manifold M^d $(d < D)$.
- Kernel eigenmap methods map the vectors in X to d-dimensional coordinates $Y = \{y_i\}_{i=1}^{N} \subset \mathbb{R}^d$ that preserve the underlying geometry of M^d.



Introduction to kernel eigenmap methods

- Given a high dimensional data set $X = \{x_i\}_{i=1}^N \subset \mathbb{R}^D$, we assume X lies on a low dimensional manifold M^d $(d < D)$.
- Kernel eigenmap methods map the vectors in X to d-dimensional coordinates $Y = \{y_i\}_{i=1}^N \subset \mathbb{R}^d$ that preserve the underlying geometry of M^d.



Basics of kernel eigenmap methods

The main components of kernel eigenmap methods are:

- Construction of an $N \times N$ symmetric, positive semi-definite kernel K,
$$K_{i,j} = K(x_i, x_j).$$
- Diagonalization of K and then choosing $d \leq D$ significant orthogonal eigenvectors of K, denoted by $v_1, \ldots, v_d$.
- Map each $x_i \in X$ to the d-dimensional vector y_i given by:
$$y_i = (v_1(i), \ldots, v_d(i)).$$

Norbert Wiener Center for Harmonic Analysis and Applications

Kernel construction

The main components of constructing the kernel K are the following:

- Consider the points $X = \{x_i\}_{i=1}^N \subset \mathbb{R}^D$ as nodes in a graph.
- Define a metric $\rho : X \times X \longrightarrow \mathbb{R}^+$, for example,

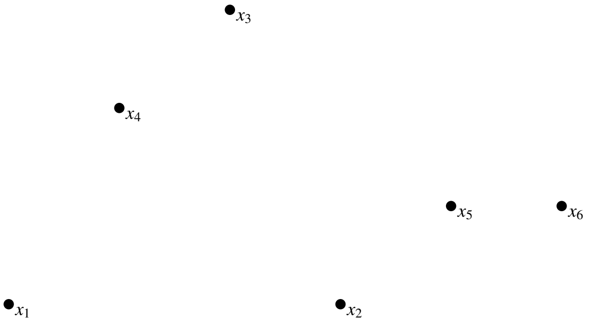
$$\rho(x_i, x_j) = \|x_i - x_j\|_{\ell^2}$$

  is the Euclidean distance.
- For each $x_i \in X$, determine the k-nearest neighbors of x_i under the metric ρ (excluding x_i). Call this set $N_k'(x_i)$.
- For each $x_j \in N_k'(x_i)$, place a (directed) edge from x_i to x_j.
- Compute weights $W = \{w_{i,j}\}_{i,j=1}^N$ on the edges of the (directed) graph.
- The kernel K is defined in terms of the weights W.

Directed graph construction

Example:

- $X = \{x_1, x_2, x_3, x_4, x_5, x_6\} \subset \mathbb{R}^2$
- Euclidean distance
- 2-nearest neighbors

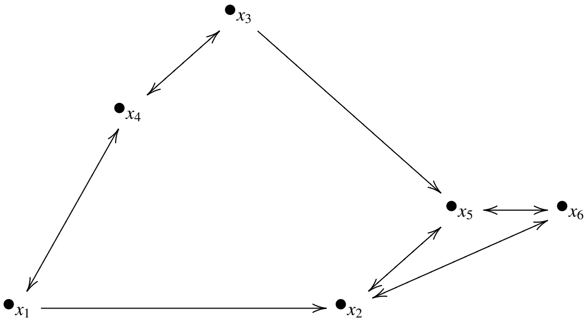


Norbert Wiener Center for Harmonic Analysis and Applications

Directed graph construction

Example:

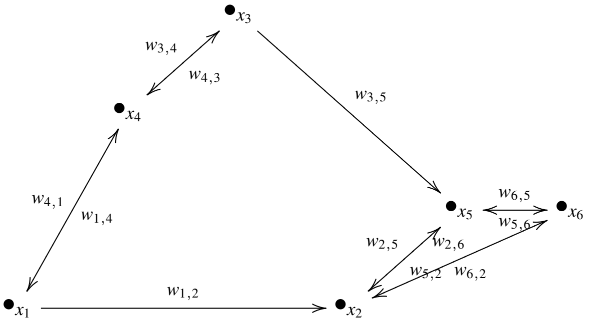
- $X = \{x_1, x_2, x_3, x_4, x_5, x_6\} \subset \mathbb{R}^2$
- Euclidean distance
- 2-nearest neighbors



Directed graph construction

Example:

- $X = \{x_1, x_2, x_3, x_4, x_5, x_6\} \subset \mathbb{R}^2$
- Euclidean distance
- 2-nearest neighbors

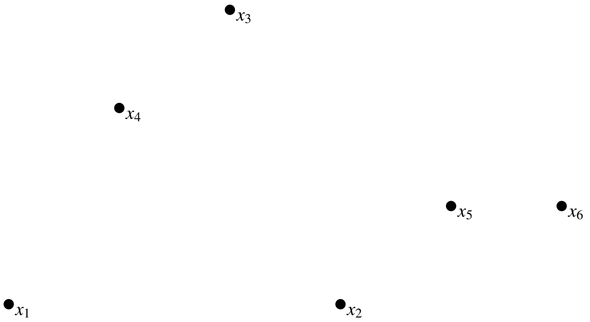


Norbert Wiener Center for Harmonic Analysis and Applications

Undirected graph construction

Example:

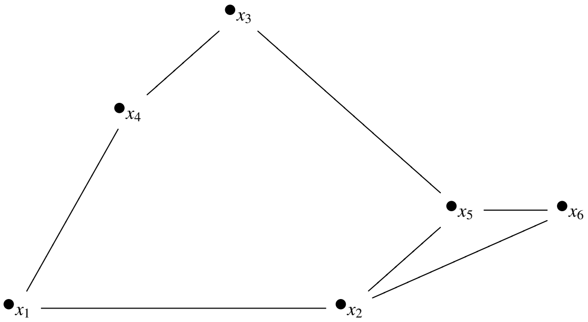
- $X = \{x_1, x_2, x_3, x_4, x_5, x_6\} \subset \mathbb{R}^2$
- Euclidean distance
- 2-nearest neighbors



Undirected graph construction

Example:

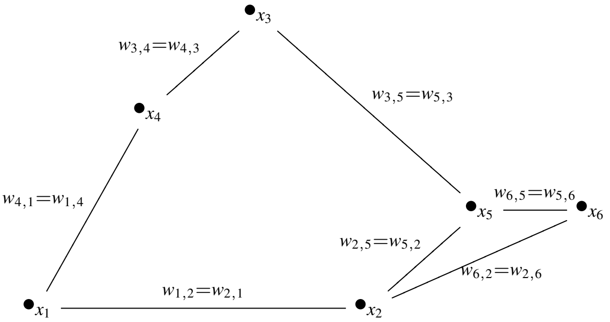
- $X = \{x_1, x_2, x_3, x_4, x_5, x_6\} \subset \mathbb{R}^2$
- Euclidean distance
- 2-nearest neighbors



Undirected graph construction

Example:

- $X = \{x_1, x_2, x_3, x_4, x_5, x_6\} \subset \mathbb{R}^2$
- Euclidean distance
- 2-nearest neighbors



Laplacian eigenmaps

We use a Laplacian eigenmap [Belkin, Niyogi; 2002] kernel K, constructed in the following manner:

- For each $x_i \in X$, determine the k-nearest neighbors of x_i under the **Euclidean distance** (excluding x_i). Call this set $N_k'(x_i)$.
- For each $x_j \in N_k'(x_i)$, place an **undirected** edge from x_i to x_j.
- For each of the edges, define symmetric weights as:

$$w_{i,j} = w_{j,i} = \exp(-\|x_i - x_j\|_{\ell^2}^2/\sigma)$$

- Set the remaining weights equal to zero, giving $W = (w_{i,j})_{i,j=1}^N$
- Set

$$K = D - W,$$

where $D_{i,i} = \sum_{j=1}^N w_{i,j}$ and $D_{i,j} = 0$ for all $i \neq j$.

Frame theory

Definition

1. $\Phi = \{\varphi_i\}_{i=1}^s$ is a frame for $\mathbb{R}^d$ if there exists $A, B > 0$ such that

$$A\|y\|^2 \leq \sum_{i=1}^{s} |\langle y, \varphi_i \rangle|^2 \leq B\|y\|^2, \quad \forall\, y \in \mathbb{R}^d.$$

2. For a frame $\Phi = \{\varphi_i\}_{i=1}^s$, the frame operator $S : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is

$$S(y) = \sum_{i=1}^{s} \langle y, \varphi_i \rangle \varphi_i.$$

- If $A = B$ and $\|\varphi_i\|_{\ell^2} = 1$ for all $i = 1, \ldots s$, then Φ is a finite unit norm tight frame (FUNTF).
- If Φ is a FUNTF, then $S(y) = \frac{s}{d} y$.

Frames and multispectral data

- We wish to use a data dependent frame $\Phi = \{\varphi_i\}_{i=1}^s$ to represent the reduced dimension space $Y = \{y_i\}_{i=1}^N \subset \mathbb{R}^d$:

$$y_i = \sum_{j=1}^{s} c_{i,j}\varphi_j, \quad \forall\, y_i \in Y.$$

- We will use a modified frame potential [Benedetto, Fickus; 2003] algorithm to construct the frames.
- Possible frame coefficients:
  - $c_{i,j} = \langle y_i, S^{-1}(\varphi_j)\rangle$
  - $c_{i,\cdot} = \arg\min_c \|c\|_{\ell^1}$ subject to $\Phi c = y_i$

Norbert Wiener Center for Harmonic Analysis and Applications

Why use frames?

Why use frames?

- Overestimating the number of classes allows for flexibility in representing mixtures and pure elements.
- Empirical evidence suggests that distinct classes are not orthogonal to each other. Unlike the orthogonal eigenvectors of K, frame elements need not be orthogonal.

Frame potential

Let $\Psi = \{\psi_i\}_{i=1}^{s} \in S^{d-1} \times \cdots \times S^{d-1}$, and consider the **frame potential** function:

$$FP(\Psi) = \sum_{i=1}^{s} \sum_{j=1}^{s} |\langle \psi_i, \psi_j \rangle|^2.$$

Theorem

Let $s > d$. The minimum value of the frame potential, FP, is s^2/d, and the minimizers are precisely the FUNTFs of s elements for $\mathbb{R}^d$.

- We use a modified frame potential to construct data (i.e. Y) dependent pseudo FUNTFs.

Norbert Wiener Center for Harmonic Analysis and Applications

Optimization problem: maximal separation

Goal: Construct a FUNTF $\Phi = \{\varphi_i\}_{i=1}^{s}$ such that each φ_i is associated to only one classifiable material.

For $\{\theta_i\}_{i=1}^{s} \in S^{d-1} \times \cdots \times S^{d-1}$, define the penalty term

$$p(\theta_i) = \sum_{j=1}^{N} |\langle y_j, \theta_i \rangle|$$

and consider the maximal separation

$$\sup_{\{\theta_i\}_{i=1}^{s}} \min\{|p(\theta_j) - p(\theta_k)| : j \neq k\}.$$

Norbert Wiener Center for Harmonic Analysis and Applications

Optimization problem: FUNTF construction

Combine maximal separation with frame potential to construct a pseudo-FUNTF $\Phi = \{\varphi_i\}_{i=1}^s$ by solving the minimization problem:

$$\sup \left\{ \min\{|p(\theta_j) - p(\theta_k)| : j \neq k\} : \{\theta_j\} \in \{\arg\min_{\Psi} FP(\Psi)\} \right\},$$

where $\Psi = \{\psi_i\}_{i=1}^s \in S^{d-1} \times \cdots \times S^{d-1}$.

- Possible alternative to maximal separation:

$$\min\{p(\theta_j - \theta_k) : j \neq k\}$$

Norbert Wiener Center for Harmonic Analysis and Applications

Implementation

We currently have implemented the following simpler version:

$$\Phi = \arg\min_{\Psi} FP(\Psi)$$

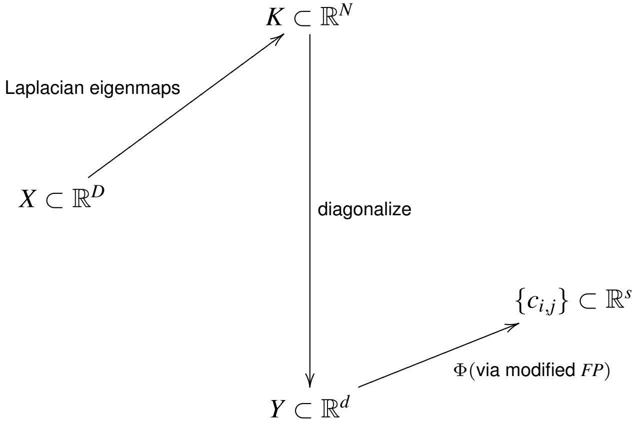
subject to

$$\sum_{i=t+1}^{s} p(\psi_i) = \sum_{i=t+1}^{s} \sum_{j=1}^{N} |\langle y_j, \psi_i \rangle| < \varepsilon,$$

for some t and ε.

Norbert Wiener Center for Harmonic Analysis and Applications

Review of algorithm

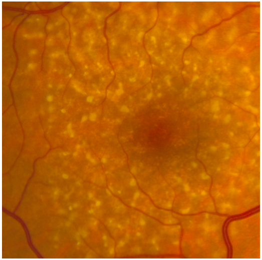


Norbert Wiener Center
for Harmonic Analysis and Applications

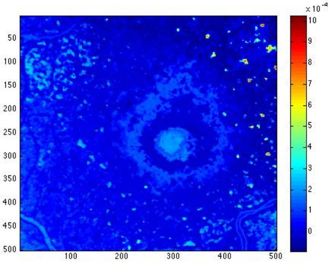
Outline

1. Introduction
   - Drusen
   - Multispectral imagery data
2. The algorithm
   - Overview
   - Kernel eigenmap methods
   - Frames
3. Results

Sparse frame coefficients

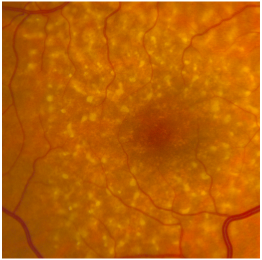


(a) Color image

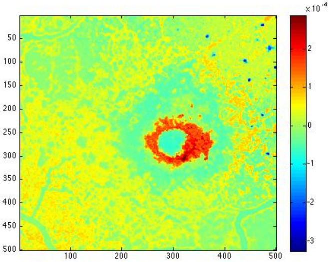


(b) Coefficient band 1

Sparse frame coefficients

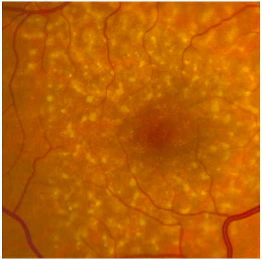


(a) Color image

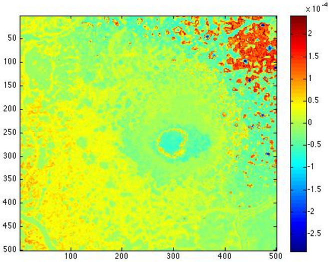


(b) Coefficient band 2

Sparse frame coefficients

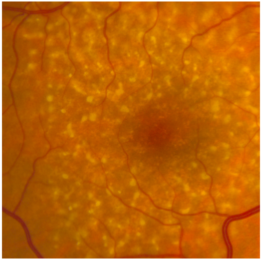


(a) Color image

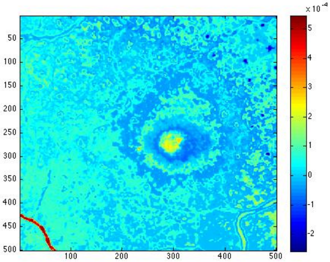


(b) Coefficient band 3

Sparse frame coefficients



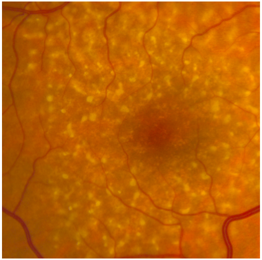
(a) Color image



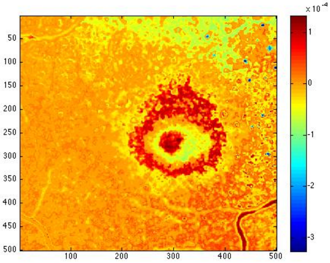
(b) Coefficient band 4

Norbert Wiener Center for Harmonic Analysis and Applications

Sparse frame coefficients

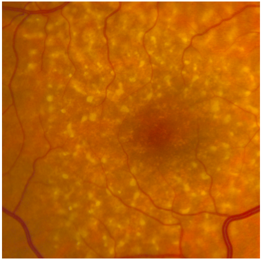


(a) Color image

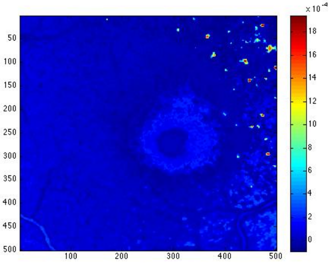


(b) Coefficient band 5

Sparse frame coefficients

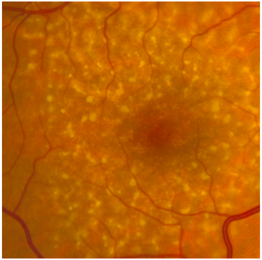


(a) Color image

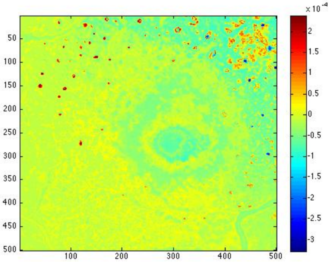


(b) Coefficient band 6

Sparse frame coefficients



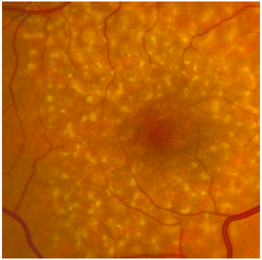
(a) Color image



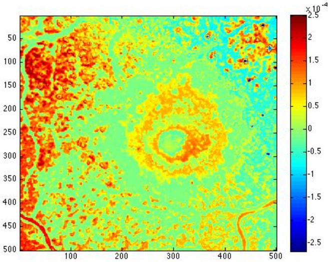
(b) Coefficient band 7

Norbert Wiener Center for Harmonic Analysis and Applications

Sparse frame coefficients



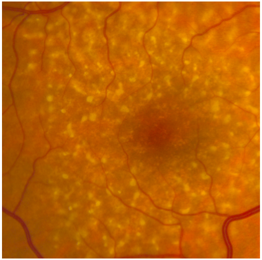
(a) Color image



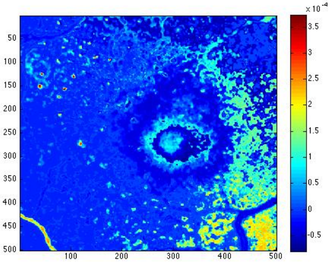
(b) Coefficient band 8

Norbert Wiener Center for Harmonic Analysis and Applications

Sparse frame coefficients

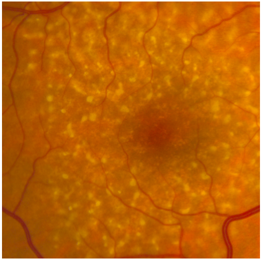


(a) Color image

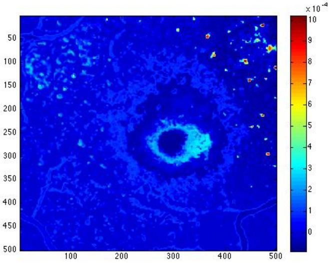


(b) Coefficient band 9

Sparse frame coefficients



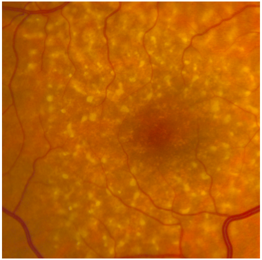
(a) Color image



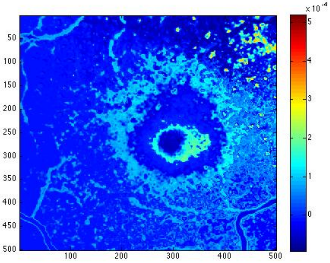
(b) Coefficient band 10

Norbert Wiener Center for Harmonic Analysis and Applications

Sparse frame coefficients

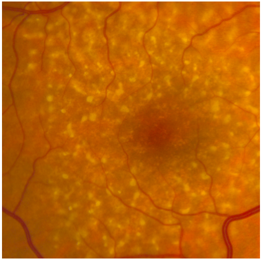


(a) Color image

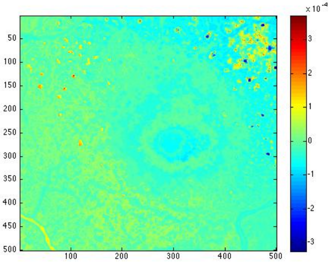


(b) Coefficient band 11

Sparse frame coefficients



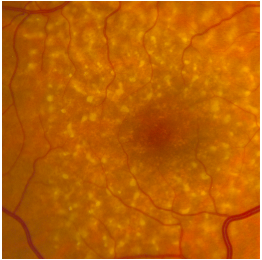
(a) Color image



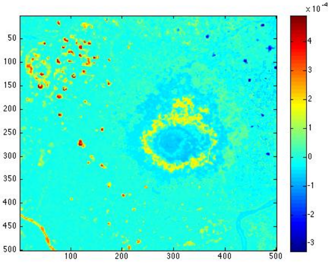
(b) Coefficient band 12

Norbert Wiener Center for Harmonic Analysis and Applications

Sparse frame coefficients

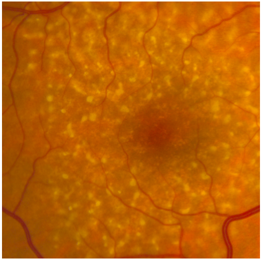


(a) Color image

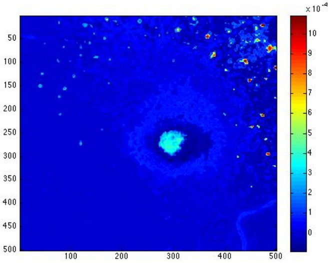


(b) Coefficient band 13

Sparse frame coefficients

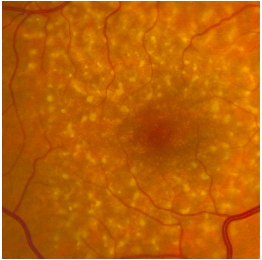


(a) Color image

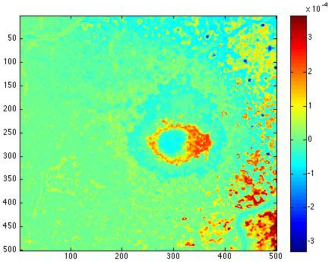


(b) Coefficient band 14

Sparse frame coefficients



(a) Color image



(b) Coefficient band 15

Thank you

Thank you for your time.

http://www.math.umd.edu/∼hirn/