Enumeration of Harmonic Frames and Frame Based Dimension Reduction

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Norbert Wiener Center for Harmonic Analysis and Applications

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Outline

Outline

Finite Frames

- Let $\Phi = \{\varphi_i\}_{i=1}^s \subset \mathbb{F}^d$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition

$\Phi = \{\varphi_i\}_{i=1}^s$ is a *finite frame* for $\mathbb{F}^d$ if there exists $A, B > 0$ such that

$$A\|y\|^2 \leq \sum_{i=1}^{s} |\langle y, \varphi_i \rangle|^2 \leq B\|y\|^2, \quad \forall\, y \in \mathbb{F}^d.$$

- If $\|\varphi_i\| = 1$ for all $i = 1, \ldots, s$, then Φ is a *unit norm frame*.
- If we can take $A = B$ in the definition, then Φ is a *tight frame*.
- If Φ is unit norm and tight, then Φ is a *finite unit norm tight frame*, or FUNTF, and we can take $A = B = s/d$.
- Φ is a frame for $\mathbb{F}^d$ if and only if $\operatorname{span}(\Phi) = \mathbb{F}^d$.

Frame Theory

Definition

For a frame $\Phi = \{\varphi_i\}_{i=1}^s$, the frame operator $S : \mathbb{F}^d \longrightarrow \mathbb{F}^d$ is

$$S(y) = \sum_{i=1}^{s} \langle y, \varphi_i \rangle \varphi_i.$$

- S is invertible.
- For each $y \in \mathbb{F}^d$, we have the following frame representation:

$$y = \sum_{i=1}^{s} \langle y, S^{-1}(\varphi_i) \rangle \varphi_i.$$

- If Φ is a FUNTF, then $S(y) = (s/d)y$ for all $y \in \mathbb{F}^d$.

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Outline

DFT-FUNTFs

- Define the $s \times s$ Discrete Fourier Transform (DFT) matrix as:

$$D_s = (e^{2\pi imn/s})_{m,n=0}^{s-1}.$$

- Define the group of integers mod s as:

$$\mathbb{Z}_s = \mathbb{Z}/s\mathbb{Z} = \{0, \ldots, s-1 \mod s\}.$$

Definition

Choose $d \leq s$ distinct columns of D_s, say $n = (n_1, \ldots, n_d) \in \mathbb{Z}_s^d$, and define $\Phi_n = \{\varphi_m : m \in \mathbb{Z}_s\}$ as

$$\varphi_m = \frac{1}{\sqrt{d}}(e^{2\pi imn_j/s})_{j=1}^{d}, \quad \forall\, m \in \mathbb{Z}_s.$$

Φ_n is a FUNTF for $\mathbb{C}^d$, and any frame of this type is called a *DFT-FUNTF*. Call $n = (n_1, \ldots, n_d)$ the *generators* of Φ_n.

Characters

- Harmonic frames are generalization of DFT-FUNTFs.
- Let G denote a finite abelian group, $G = \{g_i\}_{i=1}^s$.

Definition

A *character* of a finite abelian group G is a group homomorphism $\xi : G \longrightarrow \mathbb{C}^\times$ that satisfies

$$\xi(g_i g_j) = \xi(g_i)\xi(g_j), \quad \forall\, g_i, g_j \in G.$$

- For each $g_i \in G$, $\xi(g_i)$ is a s-th root of unity.
- There are exactly s characters, $\{\xi_i\}_{i=1}^s$.
- The *character table* of G is the matrix $(\xi_i(g_j))_{i,j=1}^s$.
- When $G \cong \mathbb{Z}_s$, the character table of G is D_s.

Harmonic Frames

- Let $\mathcal{U}(\mathbb{C}^d)$ denote the group of unitary transformations (matrices) on $\mathbb{C}^d$:

$$\mathcal{U}(\mathbb{C}^d) = \{U \in \mathcal{M}_{d\times d}(\mathbb{C}) : U^\star U = UU^\star = I_{d\times d}\}.$$

Definition

Let $\mathcal{I} \subseteq \{1, \ldots, s\}$ such that $|\mathcal{I}| = d$. Then for any $U \in \mathcal{U}(\mathbb{C}^d)$ the set

$$\Phi = \{U(\xi_i(g_j))_{i\in\mathcal{I}}\}_{j=1}^{s} \subset \mathbb{C}^d,$$

is a frame for $\mathbb{C}^d$ and is called a *harmonic frame*.

Equivalence Classes of Harmonic Frames

Definition

Two harmonic frames $\Phi = \{\varphi_i\}_{i=1}^s \subset \mathbb{C}^d$ and $\Psi = \{\psi_i\}_{i=1}^s \subset \mathbb{C}^d$ are said to be *equivalent* if there exists $U \in \mathcal{U}(\mathbb{C}^d)$ such that

$$\{\varphi_i\}_{i=1}^s = \{U\psi_i\}_{i=1}^s.$$

We denote this equivalence relation as $\Phi \sim \Psi$.

- Let $[\Phi]$ denote an equivalence class of harmonic frames with representative Φ.

Introduction to the Enumeration Problem

- **Goal:** Enumerate inequivalent harmonic frames, i.e. count the number of equivalence classes.
- **Results:** Exact, recursive formula for the number of inequivalent, prime order harmonic frames (i.e. when s is prime).
- The prime order case is simpler than the general case in part because there is only one prime order abelian group, namely $\mathbb{Z}_s$.
- This work builds upon results by:
  - Vale and Waldron (2005) - developed harmonic frames.
  - Hay and Waldron (2006) - wrote a computer program that computes all harmonic frames for a given s and d; conjectured that the number of inequivalent harmonic frames is $\mathcal{O}(s^{d-1})$.

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Setup for the Main Result

- For a fixed s (prime) and d ($d \leq s$), we backwards recursively define the set:

$$\{\gamma_c \in \mathbb{N} \cup \{0\} : c \in \mathbb{N}, c \mid s-1, \text{ and } c \mid d \text{ or } c \mid d-1\}.$$

- If $c \mid s-1$, $c \mid d$, and $c > 1$, then:

$$\gamma_c = \frac{(s-1-c)(s-1-2c)\cdots(s-1-(\frac{d}{c}-1)c)}{c^{\frac{d}{c}-1}(d/c)!} - \frac{c}{s-1}\sum_{\substack{c<b<s \\ c|b,\, b|d}} \binom{s-1}{b}\gamma_b.$$

- If $c \mid s-1$, $c \mid d-1$, and $c > 1$ then we define γ_c similarly - simply replace d with $d-1$.
- For $c = 1$, define

$$\gamma_1 = \frac{1}{s-1}\binom{s}{d} - \sum_{\substack{c|d \\ c>1}} \frac{\gamma_c}{c} - \sum_{\substack{c|d-1 \\ c>1}} \frac{\gamma_c}{c}.$$

The Main Result

Theorem

Let s be a prime number and let $1 < d < s$. Define constants γ_c as in the previous slide. The total number of inequivalent harmonic frames for $\mathbb{C}^d$ with s elements is given by:

$$\gamma_1 + \sum_{\substack{c|d \\ c>1}} \gamma_c + \sum_{\substack{c|d-1 \\ c>1}} \gamma_c.$$

Corollary

Let s be any prime number and fix d such that $1 < d < s$. Then the number of inequivalent harmonic frames for $\mathbb{C}^d$ with s elements is $\mathcal{O}(s^{d-1})$.

Outline of Proof

The proof of the theorem requires three main steps:

1. Simplify what it means for two harmonic frames to be equivalent.
2. Using the simplified notion of equivalence, bijectively map the equivalence classes $[\Phi]$ to a new space.
3. Count the objects in this new space, where it is easier to do so.

Inequivalent DFT-FUNTFs

- First note that since s is prime, every harmonic frame is derived from the character table of $\mathbb{Z}_s$, which is D_s.
- Therefore every harmonic frame is of the form $U\Phi_n$, where $U \in \mathcal{U}(\mathbb{C}^d)$ and Φ_n is a DFT-FUNTF.
- Thus we need only to find the number of inequivalent DFT-FUNTFs.

A New Equivalence Relation

- For $k \in \mathbb{Z}$, $k > 0$, let S_k denote the set of permutations on k elements.

Lemma

Let s be a prime number. If $\Phi_n = \{\varphi_m : m \in \mathbb{Z}_s\}$ and $\Psi_{n'} = \{\psi_m : m \in \mathbb{Z}_s\}$ are DFT-FUNTFs, then

$$\Phi_n \sim \Psi_{n'} \iff \begin{array}{c} \exists\ \sigma_1 \in S_s,\ \ \sigma_2 \in S_d \text{ such that} \\ \varphi_m(k) = \psi_{\sigma_1(m)}(\sigma_2(k)) \\ \forall\ m \in \mathbb{Z}_s,\ \ k = 1, \ldots, d. \end{array}$$

The Set $\mathbb{A}_s^d$

- Let $n = (n_1, \ldots, n_d) \in \mathbb{Z}_s^d$ be the generators of the DFT-FUNTF Φ_n.
- In fact, since $n_i \neq n_j$ for $i \neq j$, all such d-tuples lie in the set:

$$\tilde{\mathbb{Z}}_s^d = \{n = (n_1, \ldots, n_d) \in \mathbb{Z}_s^d : n_i \neq n_j, \ \forall\, i \neq j\}.$$

- Consider the following equivalence relation on $\tilde{\mathbb{Z}}_s^d$:

$$(n_1, \ldots, n_d) \sim (n_1', \ldots, n_d') \iff \exists \ \sigma \in S_d \ \ s.t. \ (n_1, \ldots, n_d) = (n_{\sigma(1)}', \ldots, n_{\sigma(d)}').$$

- Denote an equivalence class of $\sim$ by the representative

$$[n] = [n_1, \ldots, n_d].$$

- Define the set of all equivalence classes as $\mathbb{A}_s^d = \tilde{\mathbb{Z}}_s^d / \sim$.

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Orbits of $\mathbb{A}_s^d$

- Let $\mathbb{Z}_s^\times$ denote the group of units of $\mathbb{Z}_s$, which, when s is prime, is the set $\{1, \ldots, s-1 \mod s\}$ endowed with multiplication.
- We define the group action $\pi : \mathbb{Z}_s^\times \times \mathbb{A}_s^d \longrightarrow \mathbb{A}_s^d$ as:

$$\pi(m, [n]) = m \cdot [n] = [mn_1, \ldots, mn_d], \quad \forall\, m \in \mathbb{Z}_s^\times, \ \forall\, [n] \in \mathbb{A}_s^d.$$

- The orbits of π are the sets

$$\mathcal{O}_{[n]} = \{m \cdot [n] = [mn_1, \ldots, mn_d] : m \in \mathbb{Z}_s^\times\}.$$

- Note that the orbits $\mathcal{O}_{[n]}$ partition the set $\mathbb{A}_s^d$.

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Harmonic Frames and Orbits of $\mathbb{A}_s^d$

- Using the previous lemma, one can prove the following proposition.

Proposition

There is a one-to-one correspondence between the equivalence classes of harmonic frames and the orbits of $\mathbb{A}_s^d$ under the group action π. In particular:

$$[\Phi_n] \longleftrightarrow \mathcal{O}_{[n]}.$$

- Thus we have replaced counting the equivalence classes $[\Phi_n]$ with the problem of counting the number of orbits $\mathcal{O}_{[n]}$.

The size of $\mathcal{O}_{[n]}$

Theorem

Let s be a prime number and let $1 < d < s$. If $\mathcal{O}_{[n]}$ is an orbit of $\mathbb{A}_s^d$ under the group action π, then there exists $c \in \mathbb{N}$ such that either $c \mid d$ or $c \mid d-1$, and

$$|\mathcal{O}_{[n]}| = (s-1)/c.$$

Furthermore, the total number of orbits of order $(s-1)/c$ is given by γ_c.

- Thus the total number of orbits of $\mathbb{A}_s^d$ is given by:

$$\gamma_1 + \sum_{\substack{c|d \\ c>1}} \gamma_c + \sum_{\substack{c|d-1 \\ c>1}} \gamma_c.$$

Outline

Color Image

Red

Blue

Green

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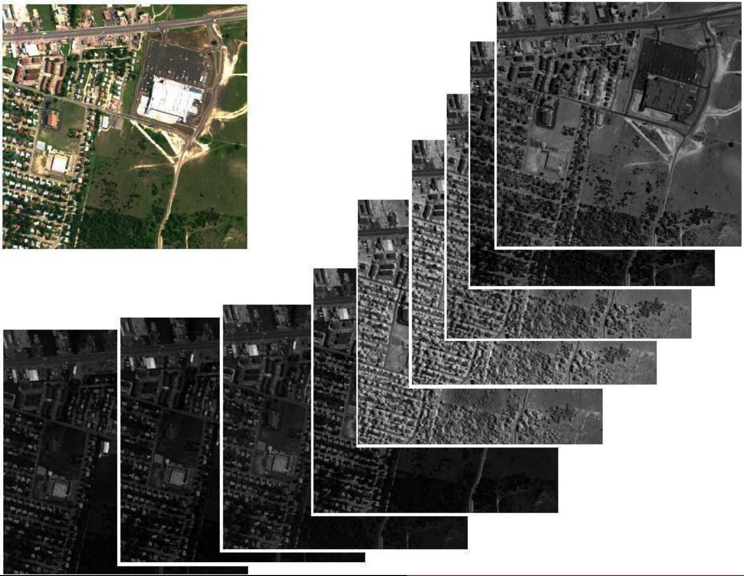
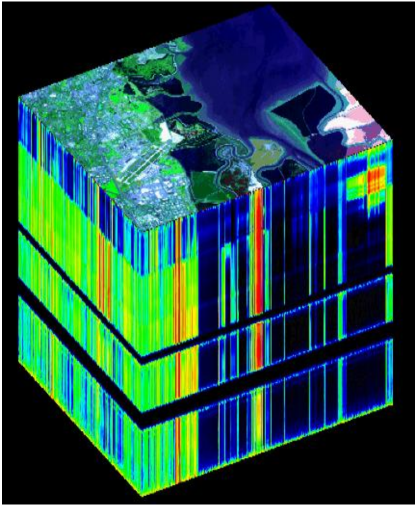
Hyperspectral Imagery Data



Hyperspectral Data Cube



Overview of Hyperspectral Imagery Data

- Hyperspectral imagery (HSI) data is characterized by the narrowness and contiguous nature of the measurements.
- HSI data sets are spectrally overdetermined, and thus provide ample spectral information to distinguish between spectrally similar (but unique) materials.
- HSI data sets can be useful for the following purposes:
  - target detection
  - material classification
  - material identification
  - mapping details of surface properties

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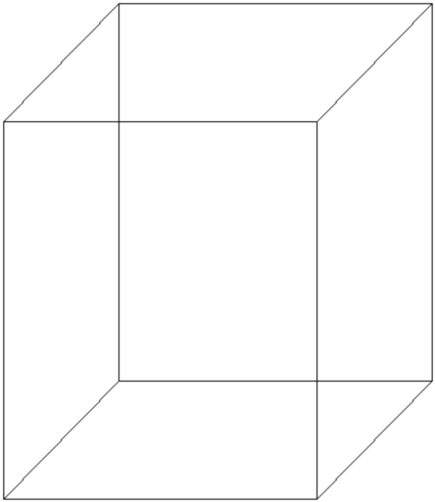
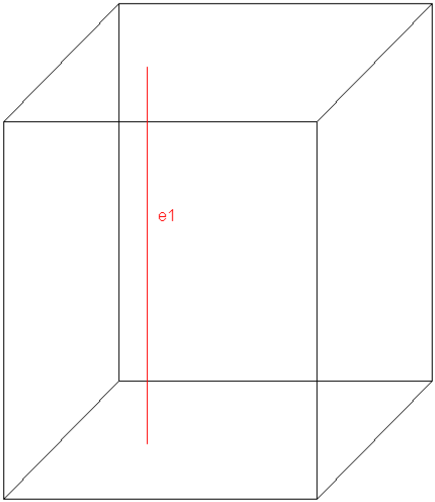
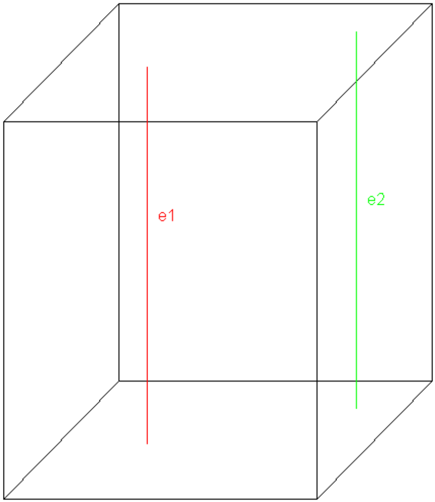
Endmember Illustration



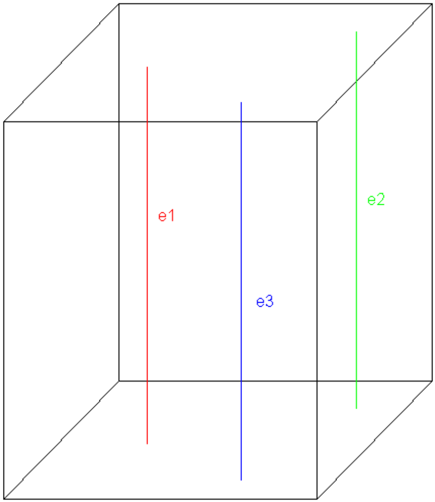
Endmember Illustration



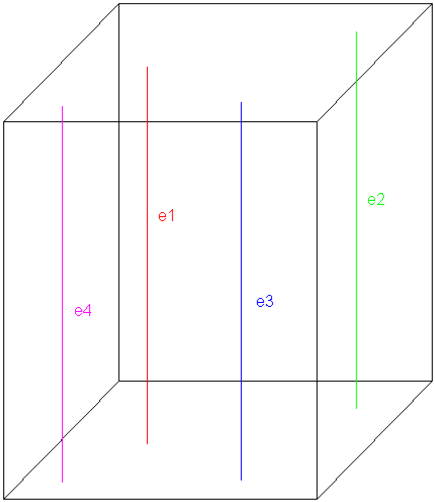
Endmember Illustration



Endmember Illustration



Endmember Illustration



Endmember Definition

- Assume our HSI data set is an $N_1 \times N_2 \times D$ cube.
  - N_1, N_2 spatial dimensions; $N = N_1 N_2$ pixels.
  - D is the spectral dimension; so D wavelengths measured.
- Let $X = \{x_i\}_{i=1}^N \subset \mathbb{R}^D$ denote the pixel vectors of the HSI data set in list form.

Definition

Endmembers are a collection of a scene's constituent spectra. If $E = \{e_i\}_{i=1}^s$ are endmembers, then the linear mixture model is

$$x_i = \sum_{j=1}^{s} \alpha_{i,j} e_j + N_{x_i}, \quad \forall\, x_i \in X,$$

where N_{x_i} is a noise vector.

Introduction to the Algorithm

We have two goals:

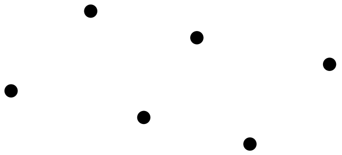
1. Map the high dimensional HSI data set X to an appropriate low dimensional space Y.
2. Represent the low dimensional space Y for the purposes of material classification.

We achieve these goals via two existing mathematical theories:

1. We use kernel eigenmap methods to determine the space Y.
2. We represent Y with an overcomplete endmember set Φ, also known as a frame.

Introduction to Kernel Eigenmap Methods

- Given a high dimensional data set $X = \{x_i\}_{i=1}^N \subset \mathbb{R}^D$, we assume X lies on a low dimensional manifold M^d $(d < D)$.
- Kernel eigenmap methods map the vectors in X to d-dimensional coordinates $Y = \{y_i\}_{i=1}^N \subset \mathbb{R}^d$ that preserve the underlying geometry of M^d.



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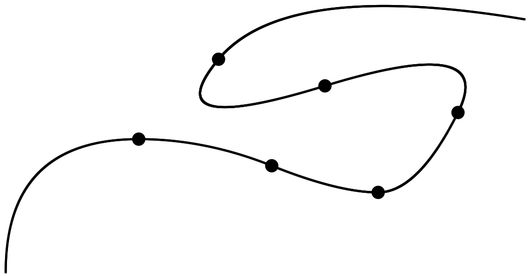


Basics of Kernel Eigenmap Methods

The main components of kernel eigenmap methods are:

- Construction of an $N \times N$ symmetric, positive semi-definite kernel K,
$$K_{i,j} = K(x_i, x_j).$$
- Diagonalization of K and then choosing $d \leq D$ significant orthogonal eigenvectors of K, denoted by $v_1, \ldots, v_d$.
- Map each $x_i \in X$ to the d-dimensional vector y_i given by:
$$y_i = (v_1(i), \ldots, v_d(i)).$$

Frames and HSI data

- We wish to use a data dependent frame $\Phi = \{\varphi_i\}_{i=1}^s$ to represent the reduced dimension space $Y = \{y_i\}_{i=1}^N \subset \mathbb{R}^d$:

$$y_i = \sum_{j=1}^{s} c_{i,j}\varphi_j, \quad \forall\, y_i \in Y.$$

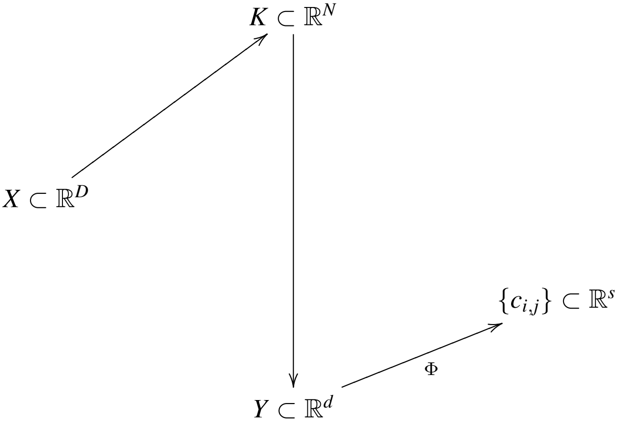
- Possible frame construction algorithms:
  - existing endmember algorithms
  - modified frame potential [Benedetto, Fickus; 2003]
- Possible frame coefficients:
  - $c_{i,j} = \langle y_i, S^{-1}(\varphi_j)\rangle$
  - $c_{i,\cdot} = \arg\min_c \|c\|_{\ell^1}$ subject to $\Phi c = y_i$

Why use frames?

Why use frames?

- Traditional endmembers may be mixtures of many prominent features.
- Overestimating the number of classes allows for flexibility in representing mixtures and pure elements.
- Empirical evidence suggests that distinct classes are not orthogonal to each other. Unlike the orthogonal eigenvectors of K, frame elements need not be orthogonal.

Review of Algorithm



Urban

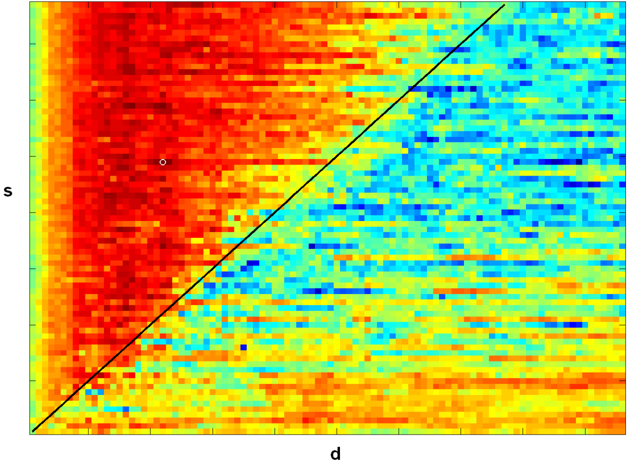
Figure: HYDICE Copperas Cove, TX – http://www.tec.army.mil/Hypercube/

Urban Classes

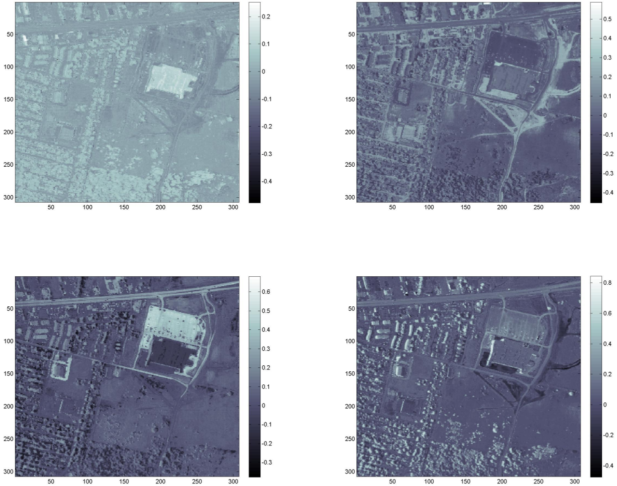
22 classes in the Urban data set, including:

- Dark asphalt
- Vegetation: grass
- Soil 1
- Soil 2
- Soil 3 (dark)
- Roof: Walmart

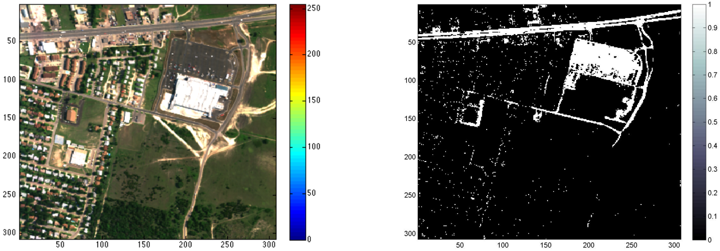
Overview of Classification Results



Sample Frame Coefficients



Dark asphalt



(e) Urban (f) Dark Asphalt

Light asphalt

(a) Urban

(b) Light asphalt

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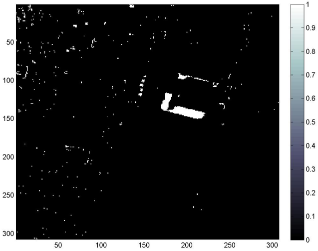
Concrete

(a) Urban



(b) Concrete

Vegetation: pasture

(a) Urban

(b) Vegetation: pasture

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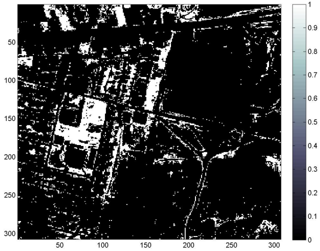
Vegetation: grass

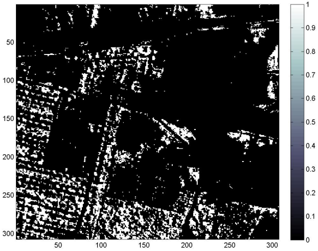
(a) Urban



(b) Vegetation: grass

Vegetation: trees

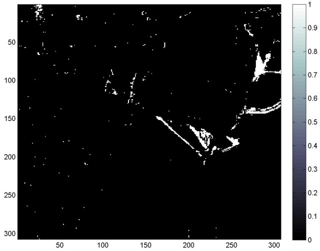
(a) Urban



(b) Vegetation: trees

Soil 1

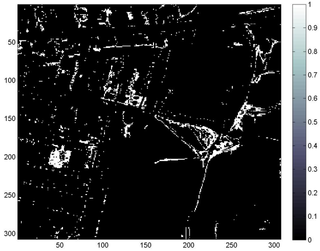
(a) Urban



(b) Soil 1

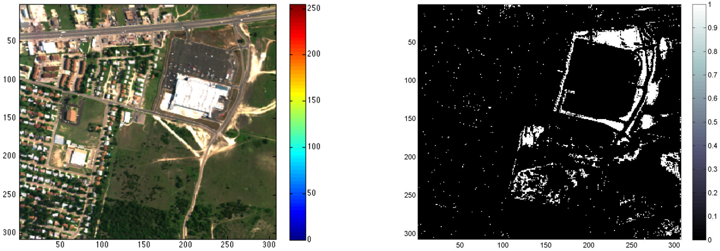
Soil 2

(a) Urban



(b) Soil 2

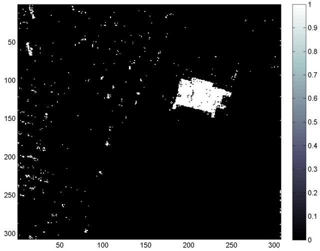
Soil 3 (dark)



(a) Urban (b) Soil 3 (dark)

Roof: Walmart

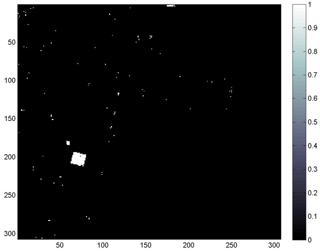
(a) Urban



(b) Roof: Walmart

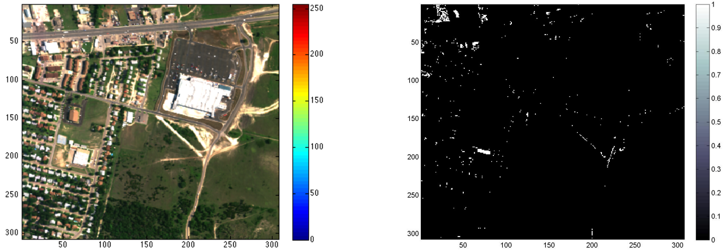
Roof: A

(a) Urban



(b) Roof: A

Roof: B

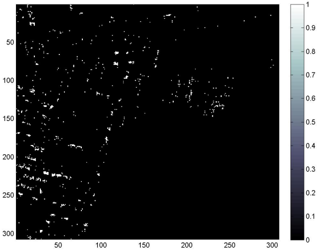


(a) Urban

(b) Roof: B

Roof: light gray

(a) Urban



(b) Roof: light gray

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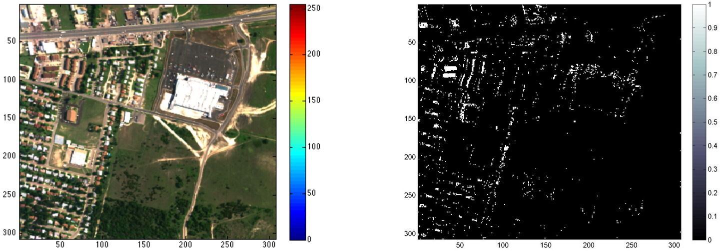
Roof: dark brown

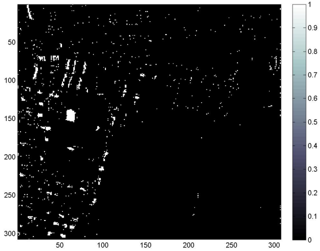


(a) Urban

(b) Roof: dark brown

Roof: church

(a) Urban



(b) Roof: church

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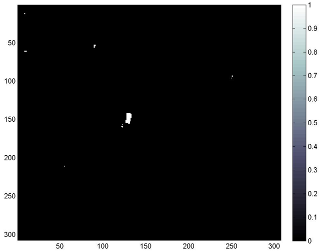
Roof: school

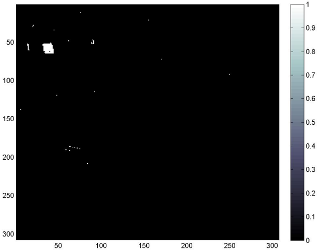
(a) Urban



(b) Roof: school

Roof: bright

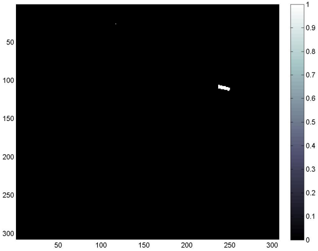
(a) Urban



(b) Roof: bright

Roof: blue green

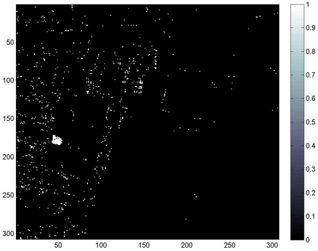
(a) Urban



(b) Roof: blue green

Tennis court

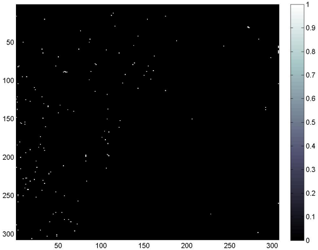
(a) Urban



(b) Tennis court

Pool water

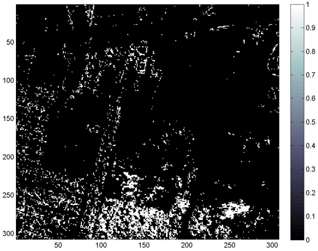
(a) Urban



(b) Pool water

Shaded vegetation

(a) Urban



(b) Shaded vegetation

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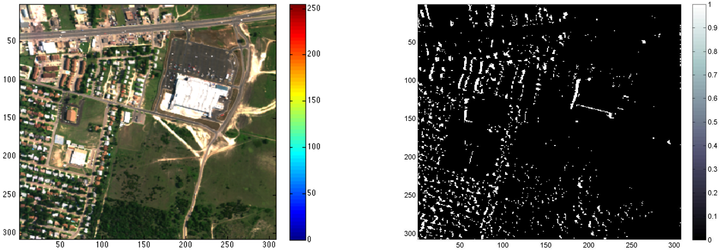
Shaded pavement



(a) Urban (b) Shaded pavement

Thank you

Thank you for your time.

http://www.math.umd.edu/∼hirn/